DRAFT VERSION JUNE 2, 2006
Preprint typeset using LATEX style emulateapj v. 11/26/04

MODIFIED GRAVITY THEORIES IN THE PALATINI APPROACH AND OBSERVATIONAL CONSTRAINTS

MICHAEL MORTONSON
University of Chicago, Department of Physics, 5640 S. Ellis Ave., Chicago, IL 60637
Draft version June 2, 2006

ABSTRACT

Theories of modified gravity provide a possible way to explain the observed late-time cosmic acceleration without dark energy. Cosmological observations can be used to constrain the parameters in these theories. Amarzguioui et al. (2005) recently used data from supernovae, baryon acoustic oscillations, and the CMB shift parameter to find constraints on parameters in an $f(R)$ theory of gravity in the Palatini approach. I describe their methods and results, and give an introduction to the necessary background on the Palatini approach to $f(R)$ gravity, modified gravity in general, and the observations used by Amarzguioui et al. (2005).

Subject headings: cosmology: theory — gravitation

1. INTRODUCTION

The observed relation between the distances and redshifts of Type Ia supernovae gives evidence that the expansion of the universe is accelerating (Riess et al. 2004). In terms of the scale factor a in the standard Friedmann-Robertson-Walker cosmology, acceleration corresponds to the condition $\ddot{a} > 0$.

Assuming that the universe is spatially homogeneous and isotropic, it can be described by the Robertson-Walker (RW) metric,

$$ds^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-\kappa r^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right], \quad (1)$$

where κ is the curvature parameter, which is zero for a flat universe, negative for an open universe, and positive for a closed universe.

The acceleration equation from general relativity, assuming that Equation (1) is the metric and that the contents of the universe can be modeled as a perfect fluid, is

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p), \quad (2)$$

where G is Newton's constant, and ρ and p are the energy density and pressure of the fluid. Assuming that ρ is non-negative, the right hand side of Equation (2) is negative for the standard components of the universe, matter (non-relativistic particles, $p = 0$) and radiation (relativistic particles, $p = \rho/3$).

This means that to get positive acceleration in this framework, we must introduce some new form of energy in the universe, which is often generically called dark energy. Observations of the cosmic microwave background power spectrum also indicate the need for dark energy, since the universe appears to have flat (or nearly flat) spatial geometry, but the combined density of matter and radiation is less than the critical density for flatness (Spergel et al. 2006).

Vacuum energy or a cosmological constant, with $p = -\rho$, could lead to the observed late-time ($z \lesssim 1$) acceleration, but this vacuum energy would have to be about a factor of 10^{120} smaller than predicted by quantum field theory (e.g., Weinberg 1989). Because no one understands why the vacuum energy should be so small, but still be nonzero, there has been a large effort to find alternative theories that could explain the cosmic acceleration. In many of these theories, a new scalar field plays the role of the dark energy. Copeland et al. (2006) present a review of the current array of dark energy theories.

Another approach to explain the cosmic acceleration is to question the assumptions in the argument for the existence of dark energy. One of these assumptions is that general relativity (GR) is the correct theory of gravity. While GR has passed a variety of tests on local (solar system) scales (Will 2006), it is possible that the theory is incomplete and that gravity on large (cosmological) scales deviates from the predictions of GR. This realization has led to an exploration of possible modifications to general relativity.

1.1. *Modified Gravity*

There are numerous ways to modify the theory of gravity. "Braneworld" models, in which the observed universe is on a 3-dimensional brane embedded in extra dimensions, can in some cases lead to non-standard gravitation at large distances (Dvali et al. 2000; Randall & Sundrum 1999a,b). Scalar-tensor theories are another possible way to modify GR, by introducing a scalar field that is non-minimally coupled to gravity (e.g., Brans & Dicke 1961; Bergmann 1968; Wagoner 1970; Damour & Esposito-Farese 1992).

The effects of gravity in cosmology can also be altered by adding extra terms to one of the relevant equations. Some of the first attempts along these lines to explain the cosmic acceleration involved adding terms to the Friedmann equation, which relates the rate of expansion of the universe to its geometry and energy density. The standard Friedmann equation, which comes from the Einstein equations for a Robertson-Walker metric, is

$$H^2 = \frac{8\pi G}{3}\rho - \frac{\kappa}{a^2}, \quad (3)$$

where $H = \dot{a}/a$ is the Hubble parameter. Freese & Lewis (2002) studied the effects of adding a term to the right hand side of Equation (3) proportional to ρ^n, where n is some arbitrary power. They found that this can lead to what they called "Cardassian expansion," *i.e.*, cosmic acceleration. Modifications to the left hand side of the Friedmann equation and their potential for causing acceleration have also been considered (Dvali & Turner 2003).

Since the Friedmann equation is derived from the Einstein equations, another approach is to modify the Einstein equations directly and see what effect that has on the Friedmann

Electronic address: mjmort@uchicago.edu

equation. In standard GR, the Einstein equations are

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{4}$$

where $G_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci tensor, R is the (Ricci) curvature scalar, and $T_{\mu\nu}$ is the energy-momentum tensor. To modify Equation (4), one could either change the form of the Einstein tensor, possibly by adding extra terms, or change how sources enter the equations on the right hand side.

However, even the Einstein equations are not really the starting point for GR, since they can be derived by varying the Einstein-Hilbert gravitational action,

$$S_g = \frac{1}{16\pi G} \int \mathrm{d}^4 x \sqrt{-g} R, \tag{5}$$

where g is the determinant of the metric. There have been several recent studies of theories that give rise to cosmic acceleration without dark energy through modification of the gravitational action. The usual approach is to replace the curvature scalar in the action, R, by some general function $f(R)$, and so these theories are often called $f(R)$ theories of gravity.

Before the discovery of the cosmic acceleration, $f(R)$ theories were investigated for their ability to produce inflation without a scalar field (inflaton). For example, theories with correction terms proportional to R^2 are known to have inflationary de Sitter space solutions at early times (Starobinsky 1980; Barrow & Ottewill 1983). Since inflation is a period of accelerated expansion in the early universe, it is natural to think that we might be able to adapt these theories to explain the acceleration observed at late times.

While the addition of terms to the gravitational action is often treated as a phenomenological approach, where one guesses a simple function of R and then works out the effects for cosmology, there are hints of a more physical motivation for the $f(R)$ theories. General relativity is viewed as a low-energy effective theory that breaks down at energies above the Planck scale, where we expect that some more complete theory of quantum gravity should apply. There are indications from studies of quantum corrections to gravity and from efforts like string theory to find a theory of quantum gravity that the full theory of gravity may have higher-order terms in the Lagrangian (e.g., Stelle 1977; Vilkovisky 1992; Nojiri & Odintsov 2003; Neupane 2006). We do not necessarily know the exact form of those extra terms, or whether they could be relevant for cosmology, but this at least suggests that considering additional terms in the action is not an entirely crazy thing to do.

There are two distinct formulations of $f(R)$ gravity, sometimes called the metric approach and the Palatini approach. A particular function $f(R)$ leads to different equations and different implications for cosmology depending upon which of these approaches is used. In the metric approach, Carroll et al. (2004) studied how $f(R)$ theories can lead to acceleration. Vollick (2003) investigated late-time acceleration from $f(R)$ theories in the Palatini approach. The metric approach to $f(R)$ theories was recently reviewed by Nojiri & Odintsov (2006), and the Palatini approach by Sotiriou & Liberati (2006).

Amarzguioui et al. (2005) (hereafter, AEMM) used a variety of cosmological observations to constrain parameters in a class of $f(R)$ theories in the Palatini approach. The goal of the present paper is to explain the work of AEMM, expanding on the relevant background material. In Section 2, I will describe the Palatini approach to $f(R)$ theories and compare it with the metric approach. Section 3 examines two specific $f(R)$ theories, including the one proposed by AEMM, and their implications for cosmology. I discuss the cosmological observations that AEMM use to constrain their $f(R)$ theory in Section 4.

Much of the work that I describe here has been done within the last year or two, and there are still several open theoretical issues. On topics about which there is currently a great deal of debate, I have attempted to present all of the differing interpretations. However, since this field is new and changing rapidly, there are probably some studies I have missed that deserve to be mentioned.

In general, I will assume units with $c = \hbar = 1$. The density parameters Ω_i will always be taken to denote their present values, while the present values of densities will be written with a subscript 0, $\rho_{i,0}$.

2. THE PALATINI APPROACH

When varying the Einstein-Hilbert action to derive the Einstein equations, the usual approach is to use the metric compatible connection, $\Gamma^\lambda_{\mu\nu}$, which means that the covariant derivative formed from the connection satisfies

$$\nabla_\rho g_{\mu\nu} = 0 \tag{6}$$

and the connection can be written in terms of $g_{\mu\nu}$:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}). \tag{7}$$

Using this connection, one varies the action with respect to the metric, assuming that the connection depends on the metric through Equation (7).

This is sometimes called the metric approach, in contrast to the Palatini approach, where one assumes that the metric and the connection are independent of each other. Although this method is generally attributed to Palatini (1919), Ferraris et al. (1982) argue that the Palatini approach as we know it was in fact invented by Einstein (1925).

In the Palatini approach, the connection that appears in the Riemann tensor is not the metric-compatible connection, so Equations (6) and (7) do not hold for this connection. Following AEMM, I will denote the affine connection, the connection used in the Riemann tensor, with a hat: $\hat{\Gamma}^\lambda_{\mu\nu}$.

In standard general relativity, both the metric approach and the Palatini approach lead to the same equations, Equation (4), so one approach seems to be just as good as the other. However, in modified gravity theories with a nonlinear $f(R)$, the metric approach and the Palatini approach lead to different results. Since the equations are the same for both approaches in standard GR, it seems reasonable to explore both approaches when we modify the theory of gravity.

2.1. *Einstein Equations*

Using the Palatini approach, the action can be varied with respect to the metric and the affine connection separately. Varying with respect to the metric gives the generalization of the Einstein equations for Palatini $f(R)$ theories.

The action is

$$S = \frac{1}{16\pi G} \int \mathrm{d}^4 x \sqrt{-g} f(R) + S_m, \tag{8}$$

where G is Newton's constant, $g = \det g_{\mu\nu}$ is the determinant of the metric, and S_m is the action for the "matter" content of the theory. (Note that I am using a different sign convention from AEMM for the gravitational action.) The matter action is assumed to be independent of the affine connection $\hat{\Gamma}$. (Although this is the usual assumption in the Palatini formalism, Sotiriou (2006) points out that dropping this assumption can have interesting and important consequences for the theory. Sotiriou & Liberati (2006) show that this does not affect the results for cosmology when the matter is taken to be a perfect fluid.)

We can separate the metric and connection dependence of the curvature scalar R by writing $R = g^{\mu\nu}R_{\mu\nu}(\hat{\Gamma})$, where the Ricci tensor $R_{\mu\nu}$ depends only on the connection and not on the metric:

$$R_{\mu\nu} = \partial_\lambda \hat{\Gamma}^\lambda_{\mu\nu} - \partial_\nu \hat{\Gamma}^\lambda_{\lambda\mu} + \hat{\Gamma}^\lambda_{\lambda\rho}\hat{\Gamma}^\rho_{\mu\nu} - \hat{\Gamma}^\lambda_{\nu\rho}\hat{\Gamma}^\rho_{\lambda\mu}. \tag{9}$$

(This also has the opposite sign to the Ricci tensor defined in AEMM.)

If we write $S = S_g + S_m$, then the variation of the gravitational term in the action, S_g, with respect to the inverse metric $g^{\mu\nu}$ is

$$\delta S_g = \frac{1}{16\pi G}\int \mathrm{d}^4x \left[-\frac{f(R)}{2\sqrt{-g}}\delta g + \sqrt{-g} f'(R)(\delta g^{\mu\nu})R_{\mu\nu}\right], \tag{10}$$

where $f'(R)$ means the derivative of $f(R)$ with respect to R. By varying the relation $\log(\det g_{\mu\nu}) = \mathrm{tr}(\log g_{\mu\nu})$, we get the variation of the determinant of the metric, $\delta g = -g g_{\mu\nu}\delta g^{\mu\nu}$ (Carroll 2004), so

$$\delta S_g = \frac{1}{16\pi G}\int \mathrm{d}^4x \sqrt{-g}\,\delta g^{\mu\nu}\left[-\frac{1}{2}f(R)g_{\mu\nu} + f'(R)R_{\mu\nu}\right]. \tag{11}$$

The energy-momentum tensor is given by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}, \tag{12}$$

so we can write the variation of the matter action S_m as

$$\begin{aligned}\delta S_m &= \int \mathrm{d}^4x \frac{\delta S}{\delta g^{\mu\nu}}\delta g^{\mu\nu} \\ &= -\frac{1}{2}\int \mathrm{d}^4x \sqrt{-g}\,\delta g^{\mu\nu} T_{\mu\nu}.\end{aligned} \tag{13}$$

Adding Equations (11) and (13) gives the variation of the total action:

$$\begin{aligned}\delta S = &\frac{1}{16\pi G}\int \mathrm{d}^4x \sqrt{-g}\,\delta g^{\mu\nu} \\ &\times \left[-\frac{1}{2}f(R)g_{\mu\nu} + f'(R)R_{\mu\nu} - 8\pi G T_{\mu\nu}\right].\end{aligned} \tag{14}$$

Finally, requiring that the variation vanishes, $\delta S = 0$, implies that the terms in square brackets must vanish, which leads to the generalized Einstein equations for $f(R)$ theories in the Palatini approach,

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{15}$$

We can recover standard GR (with a cosmological constant, Λ) by choosing $f(R) = R - 2\Lambda$, so that $f'(R) = 1$ and the Einstein equations take their usual form:

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{16}$$

In standard GR, conservation of the energy-momentum tensor implies that the left hand side of Equation (16) must satisfy the Bianchi identity,

$$\nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R. \tag{17}$$

This identity is satisfied because of the symmetries of the Riemann tensor. For $f(R)$ theories, the energy-momentum tensor should still be conserved, so we expect that a similar identity must hold for the left hand side of the generalized Einstein equations, Equation (15). Although it is not obvious that this is true, it has been shown that a generalized Bianchi identity is satisfied in general for $f(R)$ theories in both the Palatini and metric approaches (Koivisto 2005; Wang et al. 2006).

By contracting the generalized Einstein equations in Equation (15), we get a useful formula that relates the curvature scalar R and the contraction of $T_{\mu\nu}$:

$$R f'(R) - 2f(R) = 8\pi G T. \tag{18}$$

For a perfect fluid in its rest frame, assuming the RW metric, $T_{00} = \rho$ and $T_{ij} = p g_{ij}$, where ρ is the energy density of the fluid and p is its pressure. The trace of the energy-momentum tensor is then $T = g^{\mu\nu}T_{\mu\nu} = -\rho + 3p$. If Equation (18) can be solved for $R(T)$, then this tells us how to turn equations expressed in terms of R into equations written in terms of the energy density and pressure, and vice versa.

2.2. *The Connection*

We would like to use the generalized Einstein equations to do cosmology. The next step in the metric approach would be to calculate the Ricci tensor $R_{\mu\nu}$ and curvature scalar R for some particular choice of the metric, such as the Robertson-Walker metric. Then we could put the expressions for $R_{\mu\nu}$ and R into the Einstein equations and get the generalized versions of the Friedmann equations.

The problem with doing this in the Palatini approach is that the connection that appears in $R_{\mu\nu}$ (the affine connection) is no longer the same connection that can be written in terms of $g_{\mu\nu}$ using Equation (7) (the metric-compatible connection). We would like to find expressions in terms of a, ρ, and p, as in the standard Friedmann and acceleration equations. The scale factor a appears in the metric, and ρ and p are related to the curvature scalar R through Equation (18). So, if we can write the affine connection $\hat{\Gamma}$ in terms of $g_{\mu\nu}$ and R, then the generalized Einstein equations will give us relations between a, ρ, and p.

By varying the action in Equation (8) with respect to $\hat{\Gamma}$, we find

$$\hat{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + \frac{1}{2f'(R)}(\delta^\lambda_\nu \partial_\mu f' + \delta^\lambda_\mu \partial_\nu f' - g_{\mu\nu}g^{\lambda\rho}\partial_\rho f'). \tag{19}$$

Using this result with Equation (9) allows us to find the Ricci tensor in terms of $g_{\mu\nu}$ and R:

$$R_{\mu\nu} = R_{\mu\nu}(g) + \frac{3\nabla_\mu f' \nabla_\nu f'}{2(f')^2} - \frac{\nabla_\mu\nabla_\nu f'}{f'} - \frac{g_{\mu\nu}\nabla^\lambda\nabla_\lambda f'}{2f'}, \tag{20}$$

where $R_{\mu\nu}(g)$ is the Ricci tensor of the metric-compatible connection, as in AEMM. Note that ∇_μ in this expression is the covariant derivative formed from the *metric-compatible* connection Γ, not $\hat{\Gamma}$. (For the derivations of Equations (19) and (20), see the Appendix.)

After varying the action with respect to the connection, we also find that the affine connection is metric-compatible with a new metric, $h_{\mu\nu}$, which is related to the original metric by a conformal transformation (see Equation (A8)):

$$h_{\mu\nu} = f'(R)g_{\mu\nu}. \tag{21}$$

2.3. *Friedmann Equation*

We can use the Ricci tensor from Equation (20) to derive the analogue of the Friedmann equation for $f(R)$ gravity. Following AEMM, I will assume a flat Robertson-Walker metric,

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2, \tag{22}$$

where $a(t)$ is the scale factor. For this metric, the nonzero components of the metric-compatible connection are

$$\Gamma^0_{ij} = \dot{a}a\delta_{ij} \tag{23}$$

$$\Gamma^i_{0j} = \frac{\dot{a}}{a}\delta^i_j \tag{24}$$

(where the indices i and j run over the spatial components 1, 2, and 3), and the components of the Ricci tensor that depends on the metric are (Carroll 2004)

$$R_{00}(g) = -3\frac{\ddot{a}}{a} \tag{25}$$

$$R_{ij}(g) = (\ddot{a}a + 2\dot{a}^2)\delta_{ij}. \tag{26}$$

Since $f'(R)$ is a scalar, the covariant derivative of f' is just the partial derivative: $\nabla_\mu f' = \partial_\mu f'$. This means, for example, that $\nabla_\mu\nabla_\nu f' = \partial_\mu\partial_\nu f' - \Gamma^\lambda_{\mu\nu}\partial_\lambda f'$. Assuming homogeneity, R is a function of time only, so $\partial_i f' = 0$.

Consider the following combination of components of $R_{\mu\nu}(\hat{\Gamma})$:

$$\begin{aligned} R_{00} + \frac{3}{a^2}R_{11} &= R_{00}(g) + \frac{3}{a^2}R_{11}(g) + \frac{3}{2}\left(\frac{\dot{f}'}{f'}\right)^2 \\ &- \frac{1}{f'}(\ddot{f}' - \frac{3}{a^2}\Gamma^\lambda_{11}\partial_\lambda f') - \frac{1}{2f'}(g_{00} + \frac{3}{a^2}g_{11})\nabla^\lambda\nabla_\lambda f' \\ &= 6\left(\frac{\dot{a}}{a}\right)^2 + \frac{3}{2}\left(\frac{\dot{f}'}{f'}\right)^2 - \frac{\ddot{f}' - 3\dot{a}a^{-1}\dot{f}'}{f'} - \frac{-\ddot{f}' - 3a^{-2}\dot{a}a\dot{f}'}{f'} \\ &= 6H^2 + \frac{3}{2}\left(\frac{\dot{f}'}{f'}\right)^2 + 6H\frac{\dot{f}'}{f'}, \end{aligned} \tag{27}$$

where $H = \dot{a}/a$ is the Hubble parameter. Using the generalized Einstein equations from Equation (15), we can relate the expression in Equation (27) to the components of the energy-momentum tensor:

$$\begin{aligned} R_{00} + \frac{3}{a^2}R_{11} &= \frac{f}{2f'}(g_{00} + \frac{3}{a^2}g_{11}) + \frac{8\pi G}{f'}(T_{00} + \frac{3}{a^2}T_{11}) \\ &= \frac{f}{f'} + \frac{8\pi G(\rho + 3p)}{f'}. \end{aligned} \tag{28}$$

Combining Equations (27) and (28), we get the Friedmann equation for $f(R)$ gravity in the Palatini approach,

$$\left(H + \frac{\dot{f}'}{2f'}\right)^2 = \frac{1}{6f'}\left[8\pi G(\rho + 3p) + f\right]. \tag{29}$$

Equation (29) is not quite enough on its own to give us $H(R)$ because of the time derivative of $f'(R)$ on the left side of the equation. We need an expression for $\dot{R}$ in terms of R, which we can get from the continuity equation,

$$\dot{\rho} = -3H(\rho + p). \tag{30}$$

(Since this equation follows from energy-momentum conservation, $\nabla^\nu T_{\mu\nu} = 0$, it is the same as in standard GR.) The time derivative of Equation (18) is

$$\dot{R}(Rf'' - f') = 8\pi G(-\dot{\rho} + 3\dot{p}). \tag{31}$$

Substituting for $\dot{\rho}$ from Equation (30) and using $\dot{p} = (\partial p/\partial\rho)\dot{\rho}$, this becomes

$$\dot{R} = \frac{3H}{Rf'' - f'}8\pi G(\rho + p)\left(1 - 3\frac{\partial p}{\partial \rho}\right). \tag{32}$$

2.4. *Acceleration Equation*

It would be useful to have an equation for the acceleration if we are going to study theories with late-time cosmic acceleration. We can get one from the 00 component of the generalized Einstein equations. Using Equation (20) for the Ricci tensor, we find

$$\begin{aligned} R_{00} &= -3\frac{\ddot{a}}{a} + \frac{3(\dot{f}')^2}{2(f')^2} - \frac{\ddot{f}'}{f'} + \frac{1}{2f'}(-\ddot{f}' - g^{ij}\dot{a}a\delta_{ij}\dot{f}') \\ &= -3\frac{\ddot{a}}{a} + \frac{3}{2}\left[\left(\frac{\dot{f}'}{f'}\right)^2 - \frac{\ddot{f}'}{f'} - H\frac{\dot{f}'}{f'}\right]. \end{aligned} \tag{33}$$

From the generalized Einstein equations, Equation (15),

$$R_{00} = \frac{8\pi G T_{00} + f g_{00}/2}{f'} = \frac{16\pi G\rho - f}{2f'}, \tag{34}$$

so combining Equations (33) and (34) gives

$$\frac{\ddot{a}}{a} = \frac{f - 16\pi G\rho}{6f'} + \frac{1}{2}\left[\left(\frac{\dot{f}'}{f'}\right)^2 - \frac{\ddot{f}'}{f'} - H\frac{\dot{f}'}{f'}\right]. \tag{35}$$

The Hubble parameter can be written in terms of R, ρ, and p using the generalized Friedmann equation, and we can find $R(\rho, p)$ from the contracted Einstein equations, so Equation (35) gives $\ddot{a}/a$ if we know ρ and p. Recall that acceleration is defined as any period when $\ddot{a} > 0$.

2.5. *Matter-dominated Era*

Since we are looking for theories that can explain the present acceleration of the universe without dark energy, we are primarily interested in the matter-dominated case, with $p_m = 0$. From the continuity equation, Equation (30), the matter density evolves as $\rho_m = \rho_{m,0}a^{-3}$, where $\rho_{m,0}$ is the present matter density and the scale factor is normalized so that its value today is $a_0 = 1$. This implies that, during matter domination, $a = (\rho_m/\rho_{m,0})^{-1/3}$, which we can rewrite using Equation (18) with $p = 0$ as

$$a(R) = \left(\frac{8\pi G\rho_{m,0}}{2f - Rf'}\right)^{1/3} \text{ [MD]}, \tag{36}$$

where MD stands for matter domination.

An expression for $H(R)$ during matter domination follows from Equations (29) and (32), setting $p = 0$ and using $\dot{f}' = f''\dot{R}$:

$$H^2(R) = \frac{3f - Rf'}{6f'}\left[1 - \frac{3f''(Rf' - 2f)}{2f'(Rf'' - f')}\right]^{-2} \text{ [MD]}. \tag{37}$$

As in standard GR, the Hubble parameter scales as its current value, so that we can define a function $E(R)$ that is independent of the Hubble constant by $H(R) = H_0E(R)$. This can be seen by changing variables in Equation (37) from R to a dimensionless variable $\tilde{R} = R/H_0^2$. Then $H^2(R) = H_0^2H^2(\tilde{R})$, and we can identify $H(\tilde{R})$ as the function $E(R)$.

The usual critical density, $\rho_{c,0} = 3H_0^2/(8\pi G)$, is no longer the density required to make the universe flat in these $f(R)$

theories. Despite this, I will sometimes refer to the density parameter for a component i, $\Omega_i = 8\pi G\rho_i/(3H_0^2)$, following AEMM. We can still define this quantity, as long as we keep in mind that flatness does not necessarily imply that the sum of the Ω_i over all components i is 1. In particular, for the flat, matter-dominated case, $\Omega_m \neq 1$ in general.

Unless otherwise stated, I will assume in the rest of this paper (as in AEMM) a matter-dominated universe with a flat Robertson-Walker metric, so that Equations (36) and (37) hold.

2.6. *Comparison with the Metric Approach*

In the metric approach to $f(R)$ theories, where the connection used in the Riemann tensor is metric-compatible, variation of the action in Equation (8) leads to the following generalized Einstein equations:

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu f'(R) + \nabla^\lambda\nabla_\lambda f'(R)g_{\mu\nu} = 8\pi G T_{\mu\nu}. \quad \text{(metric approach)} \tag{38}$$

Notice that if $f(R) = R - 2\Lambda$, Equation (38) reduces to the standard Einstein equations, Equation (16), just as in the Palatini approach.

The metric-approach Einstein equations are identical to the Palatini-approach Einstein equations, Equation (15), except for two extra terms, each with two covariant derivatives of $f'(R)$. Since R contains second derivatives of the metric ($g_{\mu\nu}$ in the metric approach and $h_{\mu\nu}$ in the Palatini approach), the Palatini version of the Einstein equations is a second-order differential equation. The additional derivatives in Equation (38) make the metric approach Einstein equations fourth-order equations.

For some simple choices of $f(R)$, Dolgov & Kawasaki (2003) have shown that solutions to Equation (38) in the interior of some distribution of matter, like a star, are unstable and grow with time. Such theories must therefore be ruled out in the metric approach, since they are in conflict with what we observe. In the Palatini approach, Meng & Wang (2003) showed that the fact that the Einstein equations are only second-order differential equations instead of fourth-order means that these instabilities do not occur.

Solar system scale observations can also rule out certain modified gravity theories. Chiba (2003) pointed out that $f(R)$ theories in the metric approach are equivalent to scalar-tensor theories, which have a scalar field ϕ that couples to the curvature scalar R. In fact, they are equivalent to one of the earliest studied scalar-tensor theories, the Brans-Dicke theory, which has an action

$$S_{BD} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(\phi R - \frac{\omega}{\phi}\nabla^\mu\phi\nabla_\mu\phi - V(\phi)\right) + S_m, \tag{39}$$

where ω is the Brans-Dicke parameter (Brans & Dicke 1961). In more general scalar-tensor theories, ω can be a function of ϕ, but here it is assumed constant. (Actually, Equation (39) is somewhat more general than the standard Brans-Dicke theories, which do not have a potential $V(\phi)$ for the scalar field.) Since the field ϕ replaces $1/G$ in the gravitational term of the action of standard GR, the Brans-Dicke theory can be thought of as having a Newton's "constant" that varies with time (e.g., Barrow & Parsons 1997).

The Brans-Dicke theory approaches standard GR as $\omega \to \infty$, and Very Long Baseline Interferometry (VLBI) observations of the deflection by the Sun of light from radio quasars constrain the Brans-Dicke parameter to satisfy $\omega \gtrsim 3500$ (Shapiro et al. 2004; Will 2006). The metric-approach $f(R)$ theories are equivalent to Brans-Dicke theories with $\omega = 0$ (Chiba 2003), so these $f(R)$ theories are inconsistent with solar system tests of gravity. (Note, however, that the validity of this argument has been questioned by Shao et al. (2006).)

A similar analysis to that of Chiba (2003) for $f(R)$ theories in the Palatini approach shows that these theories can be rewritten as Brans-Dicke theories with $\omega = -3/2$, assuming that the matter action is independent of the affine connection (Olmo 2005; Sotiriou 2006; Wang et al. 2006). Naïvely, it would appear that the VLBI observations would rule out these theories with $\omega = -3/2$ just as they do the metric approach theories with $\omega = 0$. It turns out, though, that $\omega = -3/2$ is a pathological case in which the Brans-Dicke theory breaks down (Ni 1972; Olmo 2005). Therefore, the solar system observations cannot be applied in the same way, and it is not yet clear exactly how these observations constrain the Palatini-approach $f(R)$ theories (e.g., Wang et al. 2006; Capozziello et al. 2006b).

It may be possible to alter the metric approach theories to avoid problems with instability and solar system constraints. Nojiri & Odintsov (2003) claim that theories in which $f(R)$ contains terms with both negative and positive powers of R avoid the instability problems and could possibly satisfy solar system constraints as well. Carroll et al. (2005) suggested considering $f(R,P,Q)$ theories where the Lagrangian is a function not only of the curvature scalar but also of two other scalars, $P = R^{\mu\nu}R_{\mu\nu}$ and $Q = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$ (see also Navarro & van Acoleyen (2005)). These theories may be stable and able to satisfy solar system constraints, and De Felice et al. (2006) have shown that the parameters of the theory can be chosen so that it is free of ghost degrees of freedom and certain other problems.

2.7. *Frame-dependent Issues*

Conformal transformations between frames are often used when studying $f(R)$ theories. Sometimes it is easier to analyze a particular aspect of a theory when the equations are viewed in a certain frame. However, one has to decide which frame is the physical one, and studying a theory in the wrong frame can lead to incorrect conclusions.

As a simple example from Faraoni et al. (1998), consider the RW metric in flat space, written in terms of η, the conformal time ($d\eta = dt/a$):

$$ds^2 = a^2(\eta)(-d\eta^2 + d\mathbf{x}^2). \tag{40}$$

Now suppose we make a conformal transformation, $\tilde{g}_{\mu\nu} = a^{-2}g_{\mu\nu}$, where $g_{\mu\nu}$ is the original metric in Equation (40). Then the new metric is $\tilde{g}_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$, which is Minkowski spacetime. So, the flat RW metric is conformally related to the Minkowski metric. However, the RW metric, when used with the Einstein equations, describes an expanding universe (or at least one with a changing scale factor), while the Minkowski metric does not. The fact that these frames are related by a conformal transformation does not make them physically equivalent.

The frame we have been using so far, for example in Equation (8), is often called the Jordan frame. Flanagan (2004a) showed that, in the Palatini approach, adding an auxiliary scalar field to the action and performing a conformal transformation to another frame (called the Einstein frame) produces an action with a scalar field that has no kinetic term. (In the metric approach, this kinetic term is present.) After solving for the equation of motion of this scalar field and substituting back into the action, one finds extra terms involving the matter fields, which correspond to extensions to the standard model of particle physics that are excluded by experiments. In particular, using the Dirac action for free electrons as S_m leads to standard model corrections that are ruled out by electron-electron scattering experiments.

If this is true, the theory should be ruled out. However, Vollick (2004) argued that this conclusion is incorrect, since the matter action must added in whichever frame is physical. Flanagan (2004a) added the matter action in the Jordan frame, but pointed out the extensions to the standard model in the Einstein frame. Vollick claimed that if the matter action is added in the Jordan frame, then this is the physical frame, and it does not matter what other terms may be present in the action in some other frame. The question of whether or not $f(R)$ theories in the Palatini approach are ruled out by electron-electron scattering experiments is still controversial (Flanagan 2004b; Vollick 2005).

A similar debate exists concerning the evolution of the scale factor during matter domination in Palatini $f(R)$ theories. Through a conformal transformation from the Jordan frame to the Einstein frame which introduces a scalar field, Amendola et al. (2006a) argued that the coupling between matter and the scalar field changes the matter-dominated era so that the scale factor evolves as $a \sim t^{1/2}$ instead of the usual $a \sim t^{2/3}$. This would be in conflict with cosmological observations. The problem with this analysis, according to Capozziello et al. (2006b), is that the equations are solved to find the evolution in the Einstein frame, and then the solution is transformed back to the Jordan frame. They showed that by studying the problem only in the Jordan frame, one can obtain the standard evolution during matter domination. Again, there is so far no conclusive answer as to whether or not the scale factor evolves in a way that is inconsistent with observations (Amendola et al. 2006b). To answer these and other questions about $f(R)$ theories, much current research is concerned with issues related to conformal transformations between frames in modified gravity (e.g., Olmo & Komp 2004; Allemandi et al. 2006).

3. SPECIFIC MODIFIED GRAVITATIONAL ACTIONS

3.1. *A Simple Modified Action*

To get a sense of how the equations in the Palatini approach behave for a particular choice of $f(R)$, and also to show how acceleration can arise in these theories, I will first consider a modified action with

$$f(R) = R - \frac{\mu^4}{R}, \tag{41}$$

where μ is a parameter with dimensions of mass. This particular Lagrangian has been studied before in the metric approach by Carroll et al. (2004), and in the Palatini approach by Vollick (2003) and Meng & Wang (2003). Here, I will focus on the Palatini approach.

Assuming matter domination with a flat RW metric, Equation (18) gives a quadratic equation for $R(\rho)$ with solutions

$$R = 4\pi G\rho \pm \sqrt{3}\mu^2 \left[1 + \frac{16\pi^2}{3}\left(\frac{G\rho}{\mu^2}\right)^2\right]^{1/2}. \tag{42}$$

In the limit where the energy density is large, $G\rho/\mu^2 \gg 1$, the solutions are $R = 8\pi G\rho$ and $R = 0$. In standard GR, $R = 8\pi G\rho$ during matter domination. We do not want the theory to alter cosmology during the early matter-dominated era, when ρ is large, so we should choose the solution with the + sign in Equation (42) so that the theory has the correct behavior at early times.

In the opposite limit, where $G\rho/\mu^2 \ll 1$, the correction term in Equation (41) becomes important and we expect deviations from GR. Assuming that the universe does not recollapse, this limit should correspond to late times, since the energy density decreases as the universe expands. We can find approximate evolution equations to lowest order in this limit by letting $\rho \to 0$ in Equation (42), so that $R \approx \sqrt{3}\mu^2$.

With this approximation, $f(R) = 2\mu^2/\sqrt{3}$, $f'(R) = 4/3$, $f''(R) = -2/(3\sqrt{3}\mu^2)$, and $\dot{f}' = \ddot{f}' = 0$. From Equations (29) and (35), we get $H^2 = \mu^2/(4\sqrt{3})$ and $\ddot{a}/a = \mu^2/(4\sqrt{3})$. So, at late times, the expansion of the universe is accelerating since $\ddot{a} > 0$ (assuming $\mu^2 > 0$), and the scale factor expands exponentially since H is constant. The universe at late times in this theory approaches a de Sitter solution (Vollick 2003).

3.2. *A More General Modified Action*

The class of $f(R)$ theories studied by AEMM is given by

$$f(R) = R + \alpha H_0^2 \left(\frac{R}{H_0^2}\right)^\beta, \tag{43}$$

where α and β are dimensionless free parameters. (Due to different conventions, there is again a sign difference from AEMM here.) This choice of $f(R)$ includes ΛCDM as a special case, since $f(R) = R - 2\Lambda$ if $\beta = 0$ and $\alpha = -2\Lambda H_0^{-2} = -6\Omega_\Lambda$.

Since the power of R in the correction term is a free parameter, this choice of $f(R)$ is more general than Equation (41), which corresponds to $\beta = -1$, $\alpha = -(\mu/H_0)^4$.

Observing that $f'(R) = 1 + \alpha\beta(H_0^{-2}R)^{\beta-1} = \beta f(R)/R + 1 - \beta$, Equation (18) becomes

$$R + \alpha(2-\beta)H_0^2 \left(\frac{R}{H_0^2}\right)^\beta = 8\pi G(\rho - 3p). \tag{44}$$

For given values of α, β, and H_0, Equation (37) determines the present curvature, R_0, and then the present matter density is set by Equation (44). So Ω_m is fixed in a flat FRW universe by the choice of parameters in $f(R)$, but $\Omega_m \neq 1$ in general, as noted in Section 2.5.

3.2.1. *Early Time Behavior*

For the theory to be consistent with observed and inferred properties of the universe at early times, such as Big Bang nucleosynthesis, we can require that $f(R) \approx R$ in the early universe. (Note that this is also true in non-empty ΛCDM where $f(R) = R - 2\Lambda$, since any matter or radiation in the universe will be dominant over the cosmological constant at early times.) Equation (36) shows that $a \to 0$ as $2f - Rf' \to \infty$. If $f(R) \approx R$, then $2f - Rf' \approx R$, so "early times" correspond to large values of the curvature scalar R.

Note that this is consistent with the evolution of R in standard gravity. The curvature scalar in GR with a flat RW metric is

$$R = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] = 8\pi G(\rho - 3p). \tag{45}$$

When the universe is dominated by either matter or radiation, we have $R = 8\pi G\rho_m$ (since $\rho_r - 3p_r = 0$ and $p_m = 0$), so $R \propto a^{-3}$ during these eras.

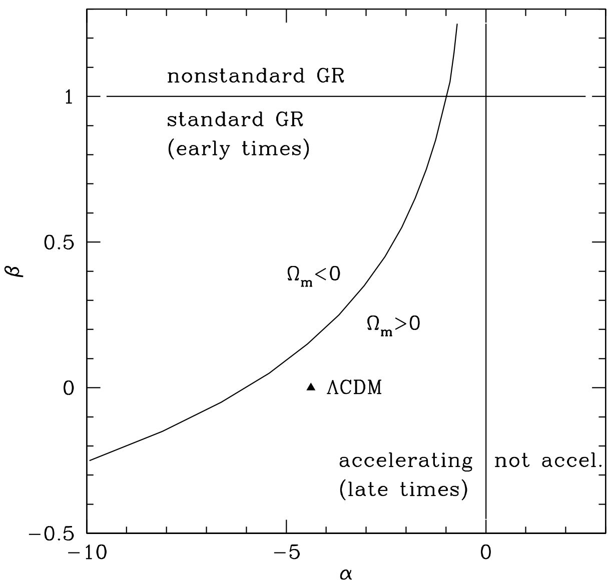

FIG. 1.— Parameter space for the $f(R)$ theory of Equation (43). The vertical line at $\alpha = 0$ separates models that have acceleration at late times from those that do not, and the horizontal line at $\beta = 1$ separates models where the theory approaches standard GR at early times from models with nonstandard gravity in the early universe. The parameter space above the curved line is not allowed because the matter density is negative in that region. The triangle marks the parameters of a theory that is equivalent to ΛCDM with $\Omega_\Lambda = 0.73$.

The $f(R)$ theory should be consistent with known properties of the early universe if $f(R) \approx R$ as $R \to \infty$. For the choice of $f(R)$ given by Equation (43), this is true as long as $\beta < 1$.

3.2.2. *Late Time Behavior*

From Equation (36), $a(R)$ has a singularity at $Rf' = 2f$. According to Equation (18), this happens when $T = 0$. This is consistent with the expectation that as $a \to \infty$ in a matter-dominated universe $T = -\rho + 3p$ should go to zero. In this limit, the curvature scalar approaches a constant,

$$R = H_0^2[-\alpha(2-\beta)]^{\frac{1}{1-\beta}}. \tag{46}$$

As with the early-time behavior, the evolution of R at late times in this $f(R)$ theory is similar to that in standard GR with a cosmological constant, Λ. In that case, from Equation (45) we have $R = 8\pi G\rho_m + 32\pi G\rho_\Lambda$, since $p_\Lambda = -\rho_\Lambda$. The matter density goes to zero at late times, and ρ_Λ is constant, so R approaches a constant in ΛCDM using standard GR. The same result follows from setting $\beta = 0$, $\alpha = -6\Omega_\Lambda$ in Equation (46) and using $\Omega_\Lambda = 8\pi G\rho_\Lambda/(3H_0^2)$.

When R is approximately constant, the Ricci tensor that depends on the affine connection is approximately equal to $R_{\mu\nu}(g)$, since the additional terms in Equation (20) all contain derivatives of R. In this limit, the 00 component of the generalized Einstein equations is

$$\left[1+\alpha\beta\left(\frac{R}{H_0^2}\right)^{\beta-1}\right]\frac{3\ddot{a}}{a} = \frac{R}{2}+\frac{\alpha H_0^2}{2}\left(\frac{R}{H_0^2}\right)^{\beta}. \tag{47}$$

Solving for $\ddot{a}/a$ with R given by Equation (46) gives

$$\frac{\ddot{a}}{a} = \frac{H_0^2}{12}[-\alpha(2-\beta)]^{\frac{1}{1-\beta}}. \tag{48}$$

Since the scale factor is always positive, the condition for the universe to be accelerating at late times ($\ddot{a} > 0$) is that the right hand side of Equation (48) must be positive. If $\beta < 1$ as required for the correct behavior at early times, then this condition is satisfied as long as $\alpha < 0$.

I will assume throughout the rest of this paper that the model parameters are restricted to $\alpha < 0$ and $\beta < 1$, as in AEMM. These constraints are illustrated in the parameter space shown in Figure 1.

3.2.3. *Other Restrictions on Parameters*

Another restriction on the parameter space for α and β comes from requiring that the present value of the matter density must be non-negative. Since the right hand side of Equation (44) is $8\pi G\rho_{m,0}$ today, this requirement implies that the left hand side evaluated today must satisfy

$$R_0 + \alpha(2-\beta)H_0^2\left(\frac{R_0}{H_0^2}\right)^{\beta} \geq 0, \tag{49}$$

where R_0 is the present value of the curvature scalar. Using Equation (37), the generalized Friedmann equation, we can find a relation between H_0, R_0, α, and β, which we can use to eliminate H_0 and R_0 in Equation (49) in favor of expressions that only depend on α and β. This turns Equation (49) into an inequality that α and β must satisfy.

Figure 1 shows the curve in parameter space corresponding to models with $\Omega_m = 0$. The region that is not allowed by the condition that the matter density must be non-negative is above and to the left of this curve. This excluded region is also shown as a shaded area in all of the figures in AEMM.

For example, if $\beta = 0$, then Equation (49) becomes $R_0 + 2H_0^2\alpha \geq 0$. Since the case $\beta = 0$ is equivalent to standard GR with a cosmological constant, we know that $H_0^2 = (8\pi G/3)(\rho_{m,0}+\rho_{\Lambda,0})$, so from Equation (44) (with p=0),

$$\begin{aligned} R_0 + 2H_0^2\alpha &= 8\pi G\rho_{m,0} \\ &= 3H_0^2 - 8\pi G\rho_{\Lambda,0} \\ &= 3H_0^2(1-\Omega_\Lambda) \\ &= H_0^2\left(3+\frac{\alpha}{2}\right) \end{aligned} \tag{50}$$

where the last line uses the fact that $\alpha = -6\Omega_\Lambda$. Since H_0^2 is always positive, the inequality of Equation (49) becomes $3+\alpha/2 \geq 0$, which implies that $\alpha \leq -6$. Writing this in terms of the cosmological constant, we require that if $\beta = 0$, $\Omega_\Lambda \leq 1$. This makes sense: in a flat universe with only matter and a cosmological constant, $\Omega_m + \Omega_\Lambda = 1$, so the matter density will be non-negative as long as $\Omega_\Lambda \leq 1$.

3.2.4. *Evolution of the Scale Factor*

As in standard cosmology, we can express the time coordinate, relative to the present time t_0, as

$$t - t_0 = \int_0^z \frac{dz'}{(1+z')H(z')}. \tag{51}$$

Since we have expressions for the redshift $z = (1/a) - 1$ and Hubble parameter as functions of R in Equations (36) and (37), we can perform the integral in Equation (51) after changing variables from z to R:

$$t - t_0 = \int_{R_0}^{R} \frac{dR'}{[1+z(R')]H(R')}\frac{\partial z}{\partial R'}, \tag{52}$$

where R_0 is the present value of the curvature scalar. Differentiating Equation (36), we find that

$$\frac{\partial z}{\partial R} = \frac{1}{3^{4/3}(\Omega_m H_0^2)^{1/3}}\frac{f' - Rf''}{(2f-Rf')^{2/3}}, \tag{53}$$

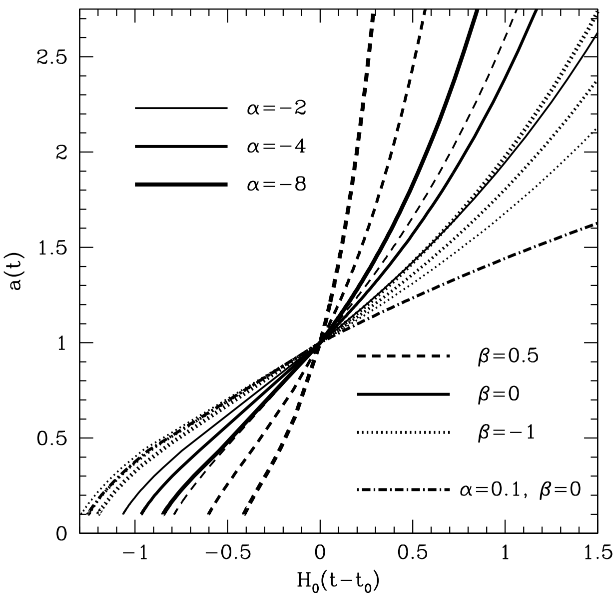

FIG. 2.— The evolution of the scale factor, $a(t)$, for the $f(R)$ theory of Equation (43) with various values of the model parameters: $\alpha = -2$ (thin lines), -4 (medium), and -8 (thick); and $\beta = 0.5$ (dashed lines), 0 (solid), and -1 (dotted). The dot-dashed curve shows a non-accelerating model with $\alpha = 0.1$ and $\beta = 0$. The scale factor today is $a(t = t_0) = 1$. Note that some of these parameter combinations are not allowed because the matter density would be negative (see Figure 1).

where I have used $8\pi G\rho_{m,0} = 3\Omega_m H_0^2$. Equation (52) can be used to find the scale factor $a = 1/(1+z)$ as a function of time for a particular choice of $f(R)$.

For $f(R)$ given by Equation (43), the evolution of the scale factor for various choices of the parameters α and β is shown in Figure 2. Note that the difference between the accelerating models ($\alpha < 0$) and the non-accelerating model ($\alpha > 0$) can be seen in the behavior of $a(t)$ for $t > t_0$.

4. OBSERVATIONS

In AEMM, four observations place constraints on the parameters α and β in the Palatini $f(R)$ theory given by Equation (43): the distance-redshift relation of type Ia supernovae, the CMB shift parameter, the baryon acoustic oscillation peak in the galaxy correlation function, and the linear growth rate of structure.

4.1. *Type Ia Supernovae*

Observations of type Ia supernovae can constrain the luminosity distance-redshift relation. AEMM use the supernovae data of Riess et al. (2004).

The luminosity distance is defined by $d_L^2 = L/(4\pi F)$, where L is the intrinsic luminosity of an object and F is the observed flux from that object. While the redshift of supernovae can be measured from their spectra, the luminosity distance cannot be directly determined from the data. Instead, one measures the apparent magnitude, m, of a supernova, which is a logarithmic version of the measured flux:

$$m = M + 5\log_{10}\left(\frac{d_L}{10\text{ pc}}\right), \tag{54}$$

where M is the absolute magnitude, the logarithmic version of the intrinsic luminosity. Distances are sometimes expressed in terms of the distance modulus, defined as $\mu = m - M$. Often the dependence on the Hubble constant is separated out by writing

$$m = M - 5\log_{10}(H_0) + 5\log_{10}\left(\frac{\tilde{d}_L}{10\text{ pc}}\right), \tag{55}$$

where $\tilde{d}_L \equiv H_0 d_L$ is independent of H_0 (e.g., Perlmutter et al. 1999). One can find constraints on cosmological parameters like Ω_m that are independent of the value of H_0 by integrating over the possible values of the Hubble constant (marginalizing over H_0).

Although supernovae are not true standard candles since they do not all have the same intrinsic luminosity, they are "standardizable" candles. This means that a set of empirical relationships among observed supernova properties can be used to find the relative distance moduli between different supernovae. The specific approach that Riess et al. (2004) use to compute distances is based on the multicolor light curve shape (MLCS) fitting method (Riess et al. 1996). This method uses supernovae light curves (the magnitude in some band observed over time) and color curves (for example, B magnitude minus V magnitude as a function of time). The light curves of brighter supernovae tend to have longer decay times, so supernovae can be calibrated to the same absolute magnitude by fitting their light curves. Also, dim supernovae are generally redder than bright supernovae during the early stages of their light curves (for about a month past the peak luminosity). These relationships are determined from well-observed low redshift supernovae with known relative distances, and then used to estimate the distances to supernovae at higher redshifts. One can determine absolute distances instead of relative distances if an absolute distance indicator, such as a Cepheid variable, is available in at least one supernova host galaxy.

In a flat FRW universe, the luminosity distance is

$$d_L(z) = (1+z)\int_0^z \frac{dz'}{H(z')}. \tag{56}$$

As in Section 3.2.4, we can change variables from z to R:

$$d_L(R) = [1+z(R)]\int_{R_0}^{R} \frac{dR'}{H(R')}\frac{\partial z}{\partial R'}, \tag{57}$$

where R_0 is the present value of the curvature.

The best fit parameters for the $f(R)$ theories considered by AEMM are $(\alpha, \beta) = (-10.0, -0.51)$ using luminosity distance-redshift constraints from the Riess et al. (2004) data and marginalizing over H_0.

4.2. *CMB Shift Parameter*

The observed position of the first acoustic peak in the cosmic microwave background (CMB) angular power spectrum, ℓ_1, can be parameterized as

$$\ell_1 = \frac{\mathcal{R}}{2}\ell_1^{(EdS)}, \tag{58}$$

where $\ell_1^{(EdS)}$ is the angular scale of the first peak in an Einstein-de Sitter universe (flat, $\Omega_m = 1$), and $\mathcal{R}$ is a constant called the CMB shift parameter (Bond et al. 1997; Efstathiou & Bond 1999; Melchiorri & Griffiths 2001). The factor of 2 in Equation (58) is included for consistency with the conventional definition of $\mathcal{R}$.

The scale of the first peak is approximately

$$\ell_1 = \frac{\pi D_A(z_r)}{s(z_r)}, \tag{59}$$

where $D_A(z_r)$ is the comoving angular diameter distance to the last scattering surface, and $s(z_r)$ is the comoving sound horizon at recombination (Efstathiou & Bond 1999). From Equation (58), we can write the shift parameter as

$$\mathcal{R} = 2\frac{s^{(EdS)}(z_r)}{s(z_r)}\frac{D_A(z_r)}{D_A^{(EdS)}(z_r)}. \tag{60}$$

The sound horizon, $s(z)$, is the distance that a sound wave can travel before redshift z. It can be written as

$$s(z) = \int_z^\infty \frac{c_s \mathrm{d}z'}{H(z')}, \tag{61}$$

where c_s is the sound speed. This is analogous to the comoving particle horizon for the propagation of light, $\eta(z) = \int_z^\infty \mathrm{d}z'/H(z')$ (in units where $c = 1$, as usual).

The speed of sound is given by $c_s^2 = \partial p/\partial\rho$ for a perfect fluid with pressure p and density ρ. At times near recombination, we need to consider both matter and radiation, so $\rho(a) = \rho_{m,0}a^{-3} + \rho_{r,0}a^{-4}$ and $p(a) = \rho_r/3 = \rho_{r,0}a^{-4}/3$. Using these expressions, we find that the sound speed squared is

$$c_s^2 = \frac{\partial p}{\partial a}\left(\frac{\partial\rho}{\partial a}\right)^{-1} = \frac{1}{3}\left(1+\frac{3\rho_m}{4\rho_r}\right)^{-1}. \tag{62}$$

It turns out that the sound horizon at recombination is relatively insensitive to the cosmological parameters. The main dependence can be described by $s(z_r) \sim 1/\sqrt{\Omega_m H_0^2}$, so $s^{(EdS)}(z_r)/s(z_r) \approx \sqrt{\Omega_m}$ (Eisenstein & White 2004). Since we are only considering $f(R)$ theories with $\beta < 1$, gravity should behave approximately the same as in standard GR for the redshifts that matter for the sound horizon, $z > z_r$.

The comoving angular diameter distance, D_A, is related to the physical angular diameter distance, d_A, by $D_A(z) = (1+z)d_A(z)$. The angular diameter distance and the luminosity distance satisfy $d_L(z) = (1+z)^2 d_A(z)$. From Equation (56), then, the comoving angular diameter distance to recombination is

$$D_A(z_r) = \int_0^{z_r} \frac{\mathrm{d}z}{H(z)}. \tag{63}$$

In the Einstein-de Sitter case, $H(z) = H_0(1+z)^{3/2}$, so

$$D_A^{(EdS)}(z_r) = 2H_0^{-1}\left(1-\frac{1}{\sqrt{1+z_r}}\right) \approx 2H_0^{-1}, \tag{64}$$

where the approximation is justified by the fact that $z_r \sim 1100$.

Combining the ratios of the sound horizons and angular diameter distances, Equation (60) becomes

$$\mathcal{R} = \sqrt{\Omega_m H_0^2}\int_0^{z_r} \frac{\mathrm{d}z}{H(z)}. \tag{65}$$

As before, we can rewrite the integral in Equation (65) in terms of the curvature scalar R:

$$\mathcal{R} = \sqrt{\Omega_m H_0^2}\int_{R_0}^{R_r} \frac{\mathrm{d}R}{H(R)}\frac{\partial z}{\partial R}, \tag{66}$$

where R_r is the value of R at recombination. (AEMM point out that by using the CMB shift parameter as written in Equation (66) we are assuming that light travels on geodesics that are associated with the metric-compatible connection, not the affine connection. Koivisto (2005) argues that this assumption is correct, but it may merit further investigation.)

According to AEMM, the value of the CMB shift parameter determined by the first year of data from the WMAP satellite is $\mathcal{R} = 1.716 \pm 0.062$, with redshift at recombination $z_r = 1088^{+1}_{-2}$ (Spergel et al. 2003). Using this value to constrain modified gravity in the Palatini approach with $f(R)$ given by Equation (43), the best fit parameters given by AEMM are $(\alpha, \beta) = (-8.4, -0.27)$.

4.3. *Baryon Acoustic Oscillations*

The oscillations of the baryon-photon plasma before recombination are imprinted not only in the anisotropies of photons in the CMB, but also in the clustering of baryons. This creates a sequence of peaks in the matter power spectrum, known as "baryon wiggles." In the two-point correlation function, which is the Fourier transform of the power spectrum, these features become a single peak (Matsubara 2004). In general, this phenomenon is usually referred to as the baryon acoustic oscillations (BAO). Since baryons only make up a small fraction of the total matter density, and the dark matter that makes up the rest is not coupled to photons, the peak in the correlation function and the baryon wiggles in the power spectrum are small and difficult to observe. Eisenstein et al. (2005) recently measured the BAO peak in the correlation function by counting pairs of luminous red galaxies (LRGs) observed in the Sloan Digital Sky Survey (SDSS).

The galaxy correlation function can be defined as $\xi(\mathbf{r}) = \langle\delta(\mathbf{x})\delta(\mathbf{x}+\mathbf{r})\rangle$, where $\delta = \delta\rho/\rho$ is the matter overdensity compared to the background density ρ. Assuming isotropy, $\xi(\mathbf{r})$ only depends on r, the magnitude of $\mathbf{r}$. One way to understand the correlation function is by considering two small volumes, dV_1 and dV_2, separated by a distance r. If the average number density of galaxies is n, then the probability that there is a galaxy in the first volume is ndV_1, and likewise for the second volume. The probability that both volumes contain a galaxy is just the product of the probabilities for the individual volumes, $n^2dV_1dV_2$, if the galaxies are distributed randomly in space. However, we know that in reality, galaxies cluster together into larger structures. To account for this, we can write the probability that there are galaxies in both dV_1 and dV_2 as $n^2dV_1dV_2[1+\xi(r)]$ (Peebles 1974).

On small scales ($r \lesssim 20$ Mpc), the correlation function is a smooth function of radius that is often modeled as a power law (Zehavi et al. 2002). The baryon acoustic oscillations create a peak in $\xi(r)$ at the scale of the sound horizon at recombination, which is around 150 Mpc (comoving).

A simple way to understand the origin of this peak is to consider an initial point perturbation in the total density (Bashinsky & Bertschinger 2001, 2002; Eisenstein et al. 2005). Dark matter remains concentrated at the location of the perturbation, but the pressure in the baryon-photon plasma causes it to expand outward in a spherical wave at the sound speed of the fluid, c_s. When the baryons and photons decouple at recombination, the baryon overdensity is left behind as a spherical shell with radius equal to the sound horizon at recombination. As the baryons and dark matter interact gravitationally, a small fraction of the dark matter is drawn into this spherical shell. In reality, of course, there is not just a single point perturbation, but the same process creates correlations in the dark matter at a scale of about 150 Mpc, which translates into galaxy correlations on this scale when structure forms.

To measure the BAO peak, Eisenstein et al. (2005) use a spherically-averaged correlation function. Measured galaxy redshifts are converted into distances using an assumed fiducial cosmology ($\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$, $h = 0.7$; $H_0 = 100h$ km s^{-1} Mpc^{-1}). Combined with the angular separations of galaxies, these line-of-sight distances can be used to derive the three-dimensional separations of galaxies. Discrepancies between the fiducial cosmology and the real cosmology shift the scale of the BAO peak. Eisenstein et al. (2005) model this shift by a single "dilation factor," $D_{V,obs}/D_{V,fid}$, where D_V is a characteristic distance to the LRG sample defined below. Comparison of the observed D_V to the theoretical prediction for the fiducial cosmology gives constraints on the combination of cosmological parameters in D_V.

In reality, observed correlations along the line of sight are different from observed correlations in the plane of the sky, since scales in the two directions depend on the cosmology in different ways (Alcock & Paczynski 1979). The comoving angular diameter distance D_A is related to a comoving scale L and its apparent angular extent on the sky θ by $D_A = L/\theta$, so an observed tangential separation depends on the cosmology through D_A as $\theta = L/D_A(z)$. A separation along the line of sight is measured as a redshift difference, Δz. Since $dD_A = dz/H(z)$ (see Equation (63)), a small difference in redshift can be expressed as $\Delta z = H(z)\Delta D_A$. The scale of the BAO peak should be independent of direction, so the measured comoving scales should be equal: $\Delta D_A = L$. Then the measured redshift difference is $\Delta z = H(z)L$, so line-of-sight separations depend on the cosmology through the Hubble parameter $H(z)$.

Since they use a spherically-averaged correlation function, Eisenstein et al. (2005) account for the Alcock-Paczynski effect by defining the comoving distance scale

$$D_V(z) = \left[\frac{z}{H(z)} D_A^2(z)\right]^{1/3}. \qquad (67)$$

This way of modeling the dependence on the cosmology can be understood by observing that the expression includes two factors of the angular diameter distance, corresponding to the two tangential directions, and one factor with $H(z)$, corresponding to the radial direction. Note that at very low redshifts, $D_A \approx z/H(z)$, so $D_V \approx D_A$. However, this is not a good enough approximation at the typical redshift of the LRG sample, $z = 0.35$.

Eisenstein et al. (2005) use their observed correlation function to constrain the parameters $D_V(z = 0.35)$ and $\Omega_m h^2$, which affects the scale of the main peak in the matter power spectrum. By combining these two, they define a parameter A that is independent of H_0 and well constrained by their observations:

$$A = \sqrt{\Omega_m H_0^2}\,\frac{D_V(z_1)}{z_1}, \qquad (68)$$

where $z_1 = 0.35$ is the typical redshift of the sample. In flat space, this parameter becomes

$$A = \sqrt{\Omega_m H_0^2}\left[\frac{1}{H(z_1)}\left(\frac{1}{z_1}\int_0^{z_1}\frac{\mathrm{d}z}{H(z)}\right)^2\right]^{1/3}. \qquad (69)$$

Again, we can write this expression in terms of R, defining R_1 as the value of R at redshift z_1:

$$A = \sqrt{\Omega_m H_0^2}\left[\frac{1}{H(R_1)}\left(\frac{1}{z(R_1)}\int_{R_0}^{R_1}\frac{\mathrm{d}R}{H(R)}\frac{\partial z}{\partial R}\right)^2\right]^{1/3}. \qquad (70)$$

The value of A determined by the BAO measurement of Eisenstein et al. (2005) is $A = 0.469 \pm 0.017$. Using just the BAO data, AEMM find that the best fit parameters for $f(R)$ given in Equation (43) are $(\alpha, \beta) = (-1.1, 0.57)$.

Combining the luminosity distance-redshift relation from SNe, the CMB shift parameter, and BAO, the best fit model is $(\alpha, \beta) = (-3.6, 0.09)$. For comparison, ΛCDM with $\Omega_\Lambda = 0.73$ corresponds to $(\alpha, \beta) = (-4.38, 0)$. Although AEMM do not explicitly give errors on the values determined for α and β, their plots of confidence contours show that within the 68% contours, α varies between about -8 and -2, and β varies between -0.3 and 0.3.

4.4. *Perturbations in Palatini Modified Gravity*

As a fourth observational constraint on the class of $f(R)$ theories defined by Equation (43), AEMM consider the linear growth rate measured by the Two Degree Field Galaxy Redshift Survey (2dFGRS) (Verde et al. 2002; Hawkins et al. 2003). The previous three constraints in Sections 4.1, 4.2, and 4.3 are mainly tests of the background evolution, but the growth rate depends on how perturbations evolve in the universe. AEMM analyze perturbations with a spherical collapse model for $f(R)$ theories based on Birkhoff's theorem, which states that the metric experienced by a particle outside a spherically symmetric distribution of matter is the same as if all the matter was located at a point at the center of the sphere (Lue et al. 2004).

The study of perturbations in modified gravity theories is still in its early stages. In scalar-tensor theories and metric-approach $f(R)$ theories, this work has been going on for several years (Hwang 1990; Hwang & Noh 1996), but perturbations have only recently been studied in the Palatini approach (Koivisto & Kurki-Suonio 2006) and for braneworld models like DGP (Sawicki & Carroll 2005; Koyama & Maartens 2006). Numerical simulations may help us better understand how structure grows when gravity is modified (Stabenau & Jain 2006).

As shown by AEMM, using the linear growth rate data makes essentially no change in the constraints on the parameters α and β. With improved theoretical understanding and better data, however, observations that depend on perturbations in $f(R)$ theories could be important for constraining these theories.

4.5. *Cosmological Constraints on Other $f(R)$ Theories*

Recently, there have been other studies similar to AEMM that use cosmological data to constrain $f(R)$ theories. In the metric approach, Mena et al. (2006) have looked at constraints from supernovae on a class of $f(R, P, Q)$ theories that have a correction term with inverse powers of R, P, and Q. Borowiec et al. (2006) analyzed Palatini $f(R)$ theories using supernovae and BAO data as in AEMM, but they assumed $f(R) \propto R^n$ for arbitrary powers of n, *without* a linear term in $f(R)$ corresponding to standard GR. Capozziello et al. (2006a) took an approach similar to Borowiec et al. (2006) with Palatini-approach $f(R)$ theories, but they considered $f(R) \propto \log R$ as well as $f(R) \propto R^n$ (again without a linear term), and their observational constraints come from supernovae and the evolution of the gas mass fractions of galaxy clusters with redshift.

5. CONCLUSIONS

Assuming that $f(R)$ theories in the Palatini approach are not ruled out by problems like those mentioned in Sections 2.6 and 2.7, the results of Amarzguioui et al. (2005) show that there exist choices of $f(R)$ that can explain the observed supernovae data without dark energy and satisfy other observational constraints from the matter correlation function and the power spectrum of the cosmic microwave background. For the particular gravity Lagrangian studied by AEMM, the data prefer parameters such that the power of R in the correction term is close to zero, making the theory very similar to ΛCDM. The simple $f(R)$ from Section 3.1, with a correction term proportional to R^{-1}, appears to be ruled out by the cosmological constraints at a high level of significance.

If we judge the ability of an observation to constrain these $f(R)$ theories by how small the error "ellipse" is in the (α,β) parameter space, then the baryon acoustic oscillations provide the tightest constraints on these parameters, followed by the supernovae data, and then the CMB shift parameter. Despite these differences, it is an important result that all three observations give constraints that are more or less consistent, especially since each observation has been cited as evidence for dark energy. If the constraints on the $f(R)$ parameters from the different tests were not consistent, it would be a sign that this class of Palatini $f(R)$ theories is insufficient to explain the data by itself.

The constraints from each observation have significant degeneracies between α and β, and none of these tests can pin down the values of α and β on its own. The degeneracies tend to lie along curves where α increases as β increases, but the slopes of these curves and the sizes of the confidence regions in different parts of the parameter space differ enough from one test to another that the joint constraints from all three tests have a much smaller degeneracy.

Since the point in the parameter space corresponding to ΛCDM lies within the 1-σ contours as shown in AEMM, the data certainly do not seem to prefer the class of modified gravity theories in Equation (43) over the simpler ΛCDM. (The best fit parameters for certain individual observational tests lie further from ΛCDM, but the errors on the parameters from a single test are quite large.) We should keep in mind that we are not seeking alternative theories because ΛCDM gives a poor fit to the data, but because we do not understand why the vacuum energy should have the observed value.

Many of the problems that are present with dark energy theories remain in $f(R)$ theories of gravity. The coincidence problem, for example, is not solved in general. The Palatini $f(R)$ theory studied by AEMM has free parameters that must be constrained by the observations, and so the coincidence problem can be rephrased in this context as the question of why the amplitude of the correction term, α, has the right value so that deviations from GR have only recently started to become important.

There are several possible directions for future work on Palatini $f(R)$ theories. Similar studies placing observational constraints on theories where $f(R)$ has a different form from that in AEMM could be interesting. Since the possible choices of $f(R)$ are limitless, it may be more useful to find ways to apply constraints from cosmological observations without first specifying the form of $f(R)$ (e.g., Capozziello et al. 2005; Multamäki & Vilja 2006).

The search for observational tests that could distinguish modified gravity from standard GR is an active area of research. There have been many recent suggestions that comparisons of distance and the growth rate of structure as a function of redshift may indicate whether gravity is modified from GR on cosmological scales (Ishak et al. 2005; Linder 2005; Bertschinger 2006). Several authors have proposed specific observations that could be sensitive to modified gravity, including the integrated Sachs-Wolfe effect in the large-scale CMB power spectrum (Hwang & Noh 2001; Zhang 2005; Sawicki & Carroll 2005), weak lensing (Knox et al. 2005; Song 2006), the matter power spectrum (Koivisto 2006), and baryon acoustic oscillations (Yamamoto et al. 2006). If these claims are correct, then observations over the next decade may provide the evidence we need to determine whether modifications to gravity could be important for cosmology.

APPENDIX

THE AFFINE CONNECTION

The equation for the affine connection $\hat{\Gamma}$ in terms of the metric-compatible connection Γ comes from varying the action (Equation (8)) with respect to $\hat{\Gamma}$. We assume that the matter action is independent of the connection, so we only need to vary the gravitational action. As in Section 2.1 we can write the curvature scalar as $R = g^{\mu\nu}R_{\mu\nu}(\hat{\Gamma})$, so the variation of S_g with respect to $\hat{\Gamma}$ is

$$\delta S_g = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}g^{\mu\nu}f'(R)\delta R_{\mu\nu}. \tag{A1}$$

The variation of $R_{\mu\nu}$ (see Equation (9)) can be written in terms of covariant derivatives of $\delta\hat{\Gamma}$:

$$\hat{\nabla}_\lambda\delta\hat{\Gamma}^\lambda_{\mu\nu} = \partial_\lambda\hat{\Gamma}^\lambda_{\mu\nu} + \hat{\Gamma}^\lambda_{\lambda\rho}\delta\hat{\Gamma}^\rho_{\mu\nu} - \hat{\Gamma}^\rho_{\lambda\mu}\delta\hat{\Gamma}^\lambda_{\rho\nu} - \hat{\Gamma}^\rho_{\lambda\nu}\delta\hat{\Gamma}^\lambda_{\mu\rho} \tag{A2}$$

$$\hat{\nabla}_\nu\delta\hat{\Gamma}^\lambda_{\lambda\mu} = \partial_\nu\hat{\Gamma}^\lambda_{\lambda\mu} + \hat{\Gamma}^\lambda_{\nu\rho}\delta\hat{\Gamma}^\rho_{\lambda\mu} - \hat{\Gamma}^\rho_{\nu\lambda}\delta\hat{\Gamma}^\lambda_{\rho\mu} - \hat{\Gamma}^\rho_{\nu\mu}\delta\hat{\Gamma}^\lambda_{\lambda\rho} \tag{A3}$$

where $\hat{\nabla}$ is the covariant derivative formed from the affine connection $\hat{\Gamma}$. (Although the connection itself is not a tensor, the difference between two connections is a tensor, and therefore the variation of the connection is also a tensor (Carroll 2004).) Using Equations (A2) and (A3), the variation of $R_{\mu\nu}$ is just $\delta R_{\mu\nu} = \hat{\nabla}_\lambda\delta\hat{\Gamma}^\lambda_{\mu\nu} - \hat{\nabla}_\nu\delta\hat{\Gamma}^\lambda_{\lambda\mu}$, so the variation of S_g is

$$\delta S_g = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}g^{\mu\nu}f'(R)(\hat{\nabla}_\lambda\delta\hat{\Gamma}^\lambda_{\mu\nu} - \hat{\nabla}_\nu\delta\hat{\Gamma}^\lambda_{\lambda\mu}). \tag{A4}$$

After integrating by parts and throwing away the surface terms by setting $\delta\hat{\Gamma}^\lambda_{\mu\nu} = 0$ on the boundary, Equation (A4) becomes

$$\begin{aligned}\delta S_g &= -\frac{1}{16\pi G}\int \mathrm{d}^4x\,[\hat{\nabla}_\lambda(\sqrt{-g}g^{\mu\nu}f')\delta\hat{\Gamma}^\lambda_{\mu\nu} - \hat{\nabla}_\nu(\sqrt{-g}g^{\mu\nu}f')\delta\hat{\Gamma}^\lambda_{\lambda\mu}] \\ &= -\frac{1}{16\pi G}\int \mathrm{d}^4x\,\delta\hat{\Gamma}^\sigma_{\alpha\mu}[\delta^\alpha_\nu\hat{\nabla}_\sigma(\sqrt{-g}g^{\mu\nu}f') - \delta^\alpha_\sigma\hat{\nabla}_\nu(\sqrt{-g}g^{\mu\nu}f')].\end{aligned} \tag{A5}$$

(Note that Stokes' theorem uses the covariant derivative formed from the affine connection, since $\hat{\Gamma}$ describes the curvature in the Palatini approach and not the metric-compatible connection Γ.)

Requiring $\delta S = 0$ implies that the expression in square brackets in the second line of Equation (A5) must vanish:

$$\delta^{\alpha}_{\nu}\hat{\nabla}_{\sigma}(\sqrt{-g}g^{\mu\nu}f') - \delta^{\alpha}_{\sigma}\hat{\nabla}_{\nu}(\sqrt{-g}g^{\mu\nu}f') = 0. \tag{A6}$$

Contracting on the indices α and σ gives

$$\hat{\nabla}_{\nu}(\sqrt{-g}g^{\mu\nu}f') = 0. \tag{A7}$$

If we define a new metric $h_{\mu\nu} \equiv f'(R)g_{\mu\nu}$, then $g^{\mu\nu} = f'h^{\mu\nu}$ and $\sqrt{-g} = (f')^{-2}\sqrt{-h}$. In terms of the new metric, Equation (A7) is

$$\hat{\nabla}_{\nu}(\sqrt{-h}h^{\mu\nu}) = 0. \tag{A8}$$

This shows that the affine connection $\hat{\Gamma}$ is metric compatible with the metric $h_{\mu\nu}$, which is related to the original metric $g_{\mu\nu}$ by a conformal transformation. We can then use Equation (7), with $\hat{\Gamma}$ replacing Γ and $h_{\mu\nu}$ replacing $g_{\mu\nu}$, to write an expression for the affine connection:

$$\begin{aligned}\hat{\Gamma}^{\lambda}_{\mu\nu} &= \frac{1}{2}h^{\lambda\rho}(\partial_{\mu}h_{\nu\rho} + \partial_{\nu}h_{\rho\mu} - \partial_{\rho}h_{\mu\nu}) \\ &= \frac{1}{2f'}g^{\lambda\rho}(f'\partial_{\mu}g_{\nu\rho} + f'\partial_{\nu}g_{\rho\mu} - f'\partial_{\rho}g_{\mu\nu} + g_{\nu\rho}\partial_{\mu}f' + g_{\rho\mu}\partial_{\nu}f' - g_{\mu\nu}\partial_{\rho}f') \\ &= \Gamma^{\lambda}_{\mu\nu} + \frac{1}{2f'}(\delta^{\lambda}_{\nu}\partial_{\mu}f' + \delta^{\lambda}_{\mu}\partial_{\nu}f' - g_{\mu\nu}g^{\lambda\rho}\partial_{\rho}f').\end{aligned} \tag{A9}$$

THE RICCI TENSOR

Equation (9) gives the Ricci tensor in terms of the affine connection, so we can use Equation (A9) to find the Ricci tensor in terms of quantities we can compute for a given metric, such as the Ricci tensor and covariant derivative formed from the metric-compatible connection.

The affine connection is the sum of the metric-compatible connection $\Gamma^{\lambda}_{\mu\nu}$ and a tensor $C^{\lambda}_{\ \mu\nu}$, so the Ricci tensor is

$$\begin{aligned}R_{\mu\nu} &= R_{\mu\nu}(g) + \partial_{\lambda}C^{\lambda}_{\ \mu\nu} - \partial_{\nu}C^{\lambda}_{\ \lambda\mu} + \Gamma^{\lambda}_{\lambda\rho}C^{\rho}_{\ \mu\nu} + C^{\lambda}_{\ \lambda\rho}\Gamma^{\rho}_{\mu\nu} + C^{\lambda}_{\ \lambda\rho}C^{\rho}_{\ \mu\nu} - \Gamma^{\lambda}_{\nu\rho}C^{\rho}_{\ \lambda\mu} - C^{\lambda}_{\ \nu\rho}\Gamma^{\rho}_{\lambda\mu} - C^{\lambda}_{\ \nu\rho}C^{\rho}_{\ \lambda\mu} \\ &= R_{\mu\nu}(g) + \nabla_{\lambda}C^{\lambda}_{\ \mu\nu} - \nabla_{\nu}C^{\lambda}_{\ \lambda\mu} + C^{\lambda}_{\ \lambda\rho}C^{\rho}_{\ \mu\nu} - C^{\lambda}_{\ \nu\rho}C^{\rho}_{\ \lambda\mu},\end{aligned} \tag{B1}$$

where $R_{\mu\nu}(g)$ is the Ricci tensor that depends on the metric-compatible connection.

The first term after $R_{\mu\nu}(g)$ in the second line of Equation (B1) is

$$\begin{aligned}\nabla_{\lambda}C^{\lambda}_{\ \mu\nu} &= -\frac{\nabla_{\lambda}f'}{2(f')^2}(\delta^{\lambda}_{\mu}\nabla_{\nu}f' + \delta^{\lambda}_{\nu}\nabla_{\mu}f' - g^{\lambda\rho}g_{\mu\nu}\nabla_{\rho}f') + \frac{1}{2f'}(2\nabla_{\mu}\nabla_{\nu}f' - g_{\mu\nu}\Box f') \\ &= -\frac{1}{(f')^2}(\nabla_{\mu}f'\nabla_{\nu}f' - \frac{1}{2}g_{\mu\nu}\nabla^{\lambda}f'\nabla_{\lambda}f') + \frac{1}{f'}(\nabla_{\mu}\nabla_{\nu}f' - \frac{1}{2}g_{\mu\nu}\Box f'),\end{aligned} \tag{B2}$$

where I am using $\partial_{\mu}f' = \nabla_{\mu}f'$ and $\nabla_{\lambda}g_{\mu\nu} = 0$, and $\Box f' = \nabla^{\mu}\nabla_{\mu}f'$. Contracting the indices λ and ν in Equation (A9) gives $C^{\lambda}_{\ \lambda\mu} = 2\nabla_{\mu}f'/f'$, so the second term after $R_{\mu\nu}(g)$ is

$$-\nabla_{\nu}C^{\lambda}_{\ \lambda\mu} = \frac{2}{(f')^2}\nabla_{\mu}f'\nabla_{\nu}f' - \frac{2}{f'}\nabla_{\mu}\nabla_{\nu}f'. \tag{B3}$$

The third term after $R_{\mu\nu}(g)$ is

$$\begin{aligned}C^{\lambda}_{\ \lambda\rho}C^{\rho}_{\ \mu\nu} &= \frac{2}{f'}\nabla_{\rho}f'C^{\rho}_{\ \mu\nu} \\ &= \frac{1}{(f')^2}(2\nabla_{\mu}f'\nabla_{\nu}f' - g_{\mu\nu}\nabla^{\lambda}f'\nabla_{\lambda}f').\end{aligned} \tag{B4}$$

Finally, the last term is

$$\begin{aligned}-C^{\lambda}_{\ \nu\rho}C^{\rho}_{\ \lambda\mu} &= -\frac{1}{4(f')^2}(\delta^{\lambda}_{\nu}\nabla_{\rho}f' + \delta^{\lambda}_{\rho}\nabla_{\nu}f' - g^{\lambda\sigma}g_{\rho\nu}\nabla_{\sigma}f')(\delta^{\rho}_{\mu}\nabla_{\lambda}f' + \delta^{\rho}_{\lambda}\nabla_{\mu}f' - g^{\rho\tau}g_{\lambda\mu}\nabla_{\tau}f') \\ &= -\frac{1}{2(f')^2}(3\nabla_{\mu}f'\nabla_{\nu}f' - g_{\mu\nu}\nabla^{\lambda}f'\nabla_{\lambda}f').\end{aligned} \tag{B5}$$

Adding these four terms together, we find the final expression for the Ricci tensor,

$$R_{\mu\nu} = R_{\mu\nu}(g) + \frac{3\nabla_{\mu}f'\nabla_{\nu}f'}{2(f')^2} - \frac{\nabla_{\mu}\nabla_{\nu}f'}{f'} - \frac{g_{\mu\nu}\nabla^{\lambda}\nabla_{\lambda}f'}{2f'}. \tag{B6}$$

REFERENCES

Alcock, C., & Paczynski, B. 1979, Nature, 281, 358

Allemandi, G., Capone, M., Capozziello, S., & Francaviglia, M. 2006, General Relativity and Gravitation, 38, 33

Amarzguioui, M., Elgaroy, O., Mota, D. F., & Multamaki, T. 2005, ArXiv Astrophysics e-prints, arXiv:astro-ph/0510519

Amendola, L., Polarski, D., & Tsujikawa, S. 2006a, ArXiv Astrophysics e-prints, arXiv:astro-ph/0603703

—. 2006b, ArXiv Astrophysics e-prints, arXiv:astro-ph/0605384

Barrow, J. D., & Ottewill, A. C. 1983, Journal of Physics A Mathematical General, 16, 2757

Barrow, J. D., & Parsons, P. 1997, Phys. Rev. D, 55, 1906

Bashinsky, S., & Bertschinger, E. 2001, Physical Review Letters, 87, 081301

—. 2002, Phys. Rev. D, 65, 123008

Bergmann, P. G. 1968, Int. J. Theor. Phys., 1, 25

Bertschinger, E. 2006, ArXiv Astrophysics e-prints, astro-ph/0604485

Bond, J. R., Efstathiou, G., & Tegmark, M. 1997, MNRAS, 291, L33

Borowiec, A., Godlowski, W., & Szydlowski, M. 2006, ArXiv Astrophysics e-prints, arXiv:astro-ph/0602526

Brans, C., & Dicke, R. H. 1961, Physical Review, 124, 925

Capozziello, S., Cardone, V. F., & Francaviglia, M. 2006a, General Relativity and Gravitation, 38, 711

Capozziello, S., Cardone, V. F., & Troisi, A. 2005, Phys. Rev. D, 71, 043503

Capozziello, S., Nojiri, S., Odintsov, S. D., & Troisi, A. 2006b, ArXiv Astrophysics e-prints, arXiv:astro-ph/0604431

Carroll, S. M. 2004, Spacetime and geometry. An introduction to general relativity (San Francisco, CA, USA: Addison Wesley)

Carroll, S. M., de Felice, A., Duvvuri, V., Easson, D. A., Trodden, M., & Turner, M. S. 2005, Phys. Rev. D, 71, 063513

Carroll, S. M., Duvvuri, V., Trodden, M., & Turner, M. S. 2004, Phys. Rev. D, 70, 043528

Chiba, T. 2003, Physics Letters B, 575, 1

Copeland, E. J., Sami, M., & Tsujikawa, S. 2006, ArXiv High Energy Physics - Theory e-prints, arXiv:hep-th/0603057

Damour, T., & Esposito-Farese, G. 1992, Classical and Quantum Gravity, 9, 2093

De Felice, A., Hindmarsh, M., & Trodden, M. 2006, ArXiv Astrophysics e-prints, arXiv:astro-ph/0604154

Dolgov, A. D., & Kawasaki, M. 2003, Physics Letters B, 573, 1

Dvali, G., Gabadadze, G., & Porrati, M. 2000, Physics Letters B, 485, 208

Dvali, G., & Turner, M. S. 2003, ArXiv Astrophysics e-prints, arXiv:astro-ph/0301510

Efstathiou, G., & Bond, J. R. 1999, MNRAS, 304, 75

Einstein, A. 1925, Sitzungber. Pruess. Akad. Wiss., 414

Eisenstein, D., & White, M. 2004, Phys. Rev. D, 70, 103523

Eisenstein, D. J. et al. 2005, ApJ, 633, 560

Faraoni, V., Gunzig, E., & Nardone, P. 1998, ArXiv General Relativity and Quantum Cosmology e-prints, arXiv:gr-qc/9811047

Ferraris, M., Francaviglia, M., & Reina, C. 1982, General Relativity and Gravitation, 14, 243

Flanagan, É. É. 2004a, Physical Review Letters, 92, 071101

—. 2004b, Classical and Quantum Gravity, 21, 3817

Freese, K., & Lewis, M. 2002, Physics Letters B, 540, 1

Hawkins, E. et al. 2003, MNRAS, 346, 78

Hwang, J.-C. 1990, Phys. Rev. D, 42, 2601

Hwang, J.-C., & Noh, H. 1996, Phys. Rev. D, 54, 1460

Hwang, J.-c., & Noh, H. 2001, Physics Letters B, 506, 13

Ishak, M., Upadhye, A., & Spergel, D. N. 2005, ArXiv Astrophysics e-prints, astro-ph/0507184

Knox, L., Song, Y. ., & Tyson, J. A. 2005, ArXiv Astrophysics e-prints, astro-ph/0503644

Koivisto, T. 2005, ArXiv General Relativity and Quantum Cosmology e-prints, gr-qc/0505128

—. 2006, Phys. Rev. D, 73, 083517

Koivisto, T., & Kurki-Suonio, H. 2006, Class. Quant. Grav., 23, 2355, astro-ph/0509422

Koyama, K., & Maartens, R. 2006, Journal of Cosmology and Astro-Particle Physics, 1, 16

Linder, E. V. 2005, Phys. Rev. D, 72, 043529

Lue, A., Scoccimarro, R., & Starkman, G. 2004, Phys. Rev. D, 69, 044005

Matsubara, T. 2004, ApJ, 615, 573

Melchiorri, A., & Griffiths, L. M. 2001, New Astronomy Review, 45, 321

Mena, O., Santiago, J., & Weller, J. 2006, Physical Review Letters, 96, 041103

Meng, X. H., & Wang, P. 2003, Classical and Quantum Gravity, 20, 4949

Multamäki, T., & Vilja, I. 2006, Phys. Rev. D, 73, 024018

Navarro, I., & van Acoleyen, K. 2005, Physics Letters B, 622, 1

Neupane, I. P. 2006, ArXiv High Energy Physics - Theory e-prints, hep-th/0605265

Ni, W.-T. 1972, ApJ, 176, 769

Nojiri, S., & Odintsov, S. D. 2003, Phys. Rev. D, 68, 123512

—. 2006, ArXiv High Energy Physics - Theory e-prints, arXiv:hep-th/0601213

Olmo, G. J. 2005, Phys. Rev. D, 72, 083505

Olmo, G. J., & Komp, W. 2004, ArXiv General Relativity and Quantum Cosmology e-prints, arXiv:gr-qc/0403092

Palatini, A. 1919, Rend. Circ. Mat. Palermo, 43, 203

Peebles, P. J. E. 1974, A&A, 32, 197

Perlmutter, S. et al. 1999, ApJ, 517, 565

Randall, L., & Sundrum, R. 1999a, Physical Review Letters, 83, 4690

—. 1999b, Physical Review Letters, 83, 3370

Riess, A. G., Press, W. H., & Kirshner, R. P. 1996, ApJ, 473, 88

Riess, A. G. et al. 2004, ApJ, 607, 665

Sawicki, I., & Carroll, S. M. 2005, ArXiv Astrophysics e-prints, astro-ph/0510364

Shao, C.-G., Cai, R.-G., Wang, B., & Su, R.-K. 2006, Physics Letters B, 633, 164

Shapiro, S. S., Davis, J. L., Lebach, D. E., & Gregory, J. S. 2004, Physical Review Letters, 92, 121101

Song, Y.-S. 2006, ArXiv Astrophysics e-prints, astro-ph/0602598

Sotiriou, T. P. 2006, ArXiv General Relativity and Quantum Cosmology e-prints, arXiv:gr-qc/0604028

Sotiriou, T. P., & Liberati, S. 2006, ArXiv General Relativity and Quantum Cosmology e-prints, arXiv:gr-qc/0604006

Spergel, D. N. et al. 2006, ArXiv Astrophysics e-prints, arXiv:astro-ph/0603449

—. 2003, ApJS, 148, 175

Stabenau, H. F., & Jain, B. 2006, ArXiv Astrophysics e-prints, astro-ph/0604038

Starobinsky, A. A. 1980, Physics Letters B, 91, 99

Stelle, K. S. 1977, Phys. Rev. D, 16, 953

Verde, L. et al. 2002, MNRAS, 335, 432

Vilkovisky, G. A. 1992, Classical and Quantum Gravity, 9, 895

Vollick, D. N. 2003, Phys. Rev. D, 68, 063510

—. 2004, Classical and Quantum Gravity, 21, 3813

—. 2005, Phys. Rev. D, 71, 044020

Wagoner, R. V. 1970, Phys. Rev. D, 1, 3209

Wang, P., Kremer, G. M., Alves, D. S. M., & Meng, X.-H. 2006, General Relativity and Gravitation, 38, 517

Weinberg, S. 1989, Reviews of Modern Physics, 61, 1

Will, C. M. 2006, Living Reviews in Relativity, 9, 3

Yamamoto, K., Bassett, B. A., Nichol, R. C., Suto, Y., & Yahata, K. 2006, ArXiv Astrophysics e-prints, astro-ph/0605278

Zehavi, I. et al. 2002, ApJ, 571, 172

Zhang, P. 2005, ArXiv Astrophysics e-prints, astro-ph/0511218