A review of:
"A new cosmic microwave background constraint to primordial gravitational waves" [1]

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I. INTRODUCTION

Gravitational waves are a mathematical prediction from Einstein's equations of general relativity – they are ripples of the space-time metric that can propagate at the speed of light, affecting the relative distances between objects through which they pass. Gravitational waves are produced in many systems, and because they travel through spacetime, unperturbed by matter, the very early universe can be probed through their detection and analysis. For example, if produced during an inflation period of the early universe, the gravitational waves would leave a signature on the cosmic microwave background radiation (CMB) polarization. Observing this expected polarization would help confirm inflation models and also the theory of general relativity.

A recent paper by Smith et al [1] considers the effects of a general cosmological gravitational wave background (CGWB) on observables today. The main focus of the paper is not to check a particular model of inflation, but to derive a new limit on the energy-density of gravitational waves over a range of frequencies ($10^{-15} - 10^{-10}$ Hz) that currently has no other constraints. They consider two types of backgrounds: one in which the gravitational waves are produced adiabatically with other particle species, and another which is non-adiabatic (as in inflation).

The first part of this paper will be an introduction to gravitational waves and how they are produced, followed by current constraints on the energy density of gravitational waves today. Then, the new constraints from [1] will be discussed and conclusions drawn about how these new constraints will affect future work.

II. GRAVITATIONAL WAVE SOLUTIONS

Given a spacetime metric, $\overline{g}_{\mu\nu}$, one can add a small perturbative tensor, $h_{\mu\nu}$, so that the full metric is now

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + h_{\mu\nu}, \tag{1}$$

where $|h_{\mu\nu}| \ll |\overline{g}_{\mu\nu}|$. Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \tag{2}$$

can be expanded using the full form $g_{\mu\nu}$. Then, to 0th order in the perturbation, the original solutions to Einstein's equations for $g_{\mu\nu}$ are recovered, and to 1st order in $h_{\mu\nu}$ new linear equations satisfied by the perturbation are derived. The case of $g_{\mu\nu} = \eta_{\mu\nu}$, the Minkowski space metric is performed in great detail in [2], but for more complicated metrics, the calculation process is the same. Under the metric perturbation, indices raised and lowered with the metric remain done using $\overline{g}_{\mu\nu}$, not using $h_{\mu\nu}$ (or that would lead to terms that are second or higher order in the perturbation). The Christoffel symbols, then the Ricci tensor, then the curvature scalar must be linearized with respect to $h_{\mu\nu}$. After the choice of the transverse-transverse gauge

$$\partial_\nu h^{\mu\nu} = 0, \quad \overline{g}^{\mu\nu} h_{\mu\nu} = 0, \tag{3}$$

the full equation describing the perturbation is found to be the massless Klein-Gordon equation:

$$(\partial_t^2 - \nabla^2) h_{ij} = 0, \tag{4}$$

so that gravitational waves are equivalent to quantum field theory of spin-2 massless graviton with two polarization states. There are two degrees of freedom left, corresponding to the two polarizations. The solutions to the Klein-Gordon equation are travelling waves of the form

$$h_{ij} = C_{ij} e^{i k_\sigma x^\sigma} \tag{5}$$

with C_{ij} a constant, symmetric, traceless tensor. The standard example is a wave travelling in the z direction so that $k = (\omega, 0, 0, \omega)$, then $C_{11} = -C_{22} = C_+$ and $C_{12} = C_{21} = C_\times$. If $C_\times = 0$, the effect of a wave passing through a circle of particles are in relative oscillatory motion as shown in Figure 1.

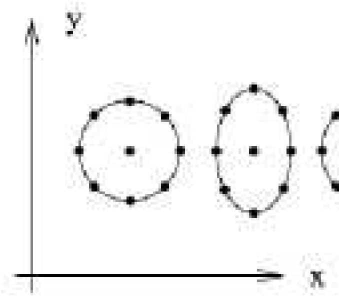

FIG. 1: Relative motion of particles as a "+" polarized gravitational wave passes though space. Figure taken from [3].

In the cosmologically relevant expanding Robertson-Walker flat universe with scale factor $a(t)$, the metric is given by

$$ds^2 = dt^2 - a^2(t)d\vec{x}^2. \tag{6}$$

Calculation of the linearized perturbation theory is completed in [4], for example.

Because passing gravitational waves move test particles relative to each other, there are large interferometer experiments, currently underway hoping to directly detect passing waves. The Laser Interferometer Space Antenna (LISA) planned to launch in 2015 will be monitoring test masses for fluctuations that could be from gravitational waves with frequencies in the range $10^{-5} - 1$ Hz.

A. Sources of Gravitational Radiation

Besides propagating through empty space, gravitational waves can be emitted from sources as gravitational radiation. However, due to the conservation of total momentum the time rate of change of the mass dipole moment must be zero, and sources with a varying dipole moment do not emit gravitational radiation. However, sources with time-varying quadrupole moments do radiate gravitational waves. One example of indirect observation of gravitational radiation is that of the pulsar star PSR 1913+16 which is part of a binary star system. Pulsar stars emit electromagnetic radiation with a regular period corresponding to its fast rotation (59 ms for PSR 1913+16), but by observing slight changes in the period, the binary stars are believed to be spiraling in towards each other as their energy is decreased by the gravitational radiation being emitted [5].

III. CURRENT GW ENERGY DENSITY CONSTRAINTS

A. Big Bang Nucleosynthesis

The number of degrees of freedom g_* at the time of nucleosynthesis is a critical component that determines the abundances of primordial light-elements and the neutron to proton ratio at freeze-out, because the scale factor at the time, $H \sim g_*^{1/2}$. If g_* is increased, the scale factor is increased, then nucleosynthesis happens earlier (since $\Gamma \sim H$ earlier), so the neutron to proton ratio is increased, and so is the ^{4}He abundance. Given the observed ^{4}He abundance or n_n/n_p then limits the number of neutrino extra degrees of freedom to about 1.4 or the energy density to about or under 8 x 10^{-6}[6]. This bound only covers a frequency range down to about 10^{-10} Hz, corresponding to the horizon size at nucleosynthesis.

B. Pulsar-timing

Pulsars which have exceptionally regular periodic EM signals can be used to detect passing gravitational waves. By watching a distant object like a pulsar one has a very long scale over which to measure gravitational waves. If a gravitational wave passes between the observer and the pulsar, the distance between the objects will be perturbed, and this will be realized as a change in observed frequency of the pulsar, with the change in the frequency proportional to the wave amplitude. By monitoring pulsars, a limit of CGWB energy density on the order of 10^{-8} Hz is found [7].

C. Cosmic Microwave Background Radiation

Finally, the anisotropy in the CMB radiation constrains the CGWB energy density at very low frequencies corresponding to the horizon size today, $H_0^{-1} \sim 10^{-16}$ Hz. All of the above constraints are shown in Figure 3 in comparison with the new constraints found below.

IV. NEW CONSTRAINTS FROM [1]

The angular power spectrum of the CMB that has been recently analyzed constrains the radiation energy density at recombination. Since the radiation energy density is from a combination of photons and neutrinos, the constraints on the total density can be written as a constraint on the effective number of neutrinos, N_{eff}. The standard model predicts 3 neutrino species, so $N_{eff} > 3$ means there is extra contribution to the energy density from another particle species. Extra relativistic energy at recombination affects the outcome of the observable universe because it will delay equivalence and the scale entering the horizon at equivalence will be bigger [8]. The energy density today corresponding to one effective neutrino species can be found from the general relation between relativistic energy density and temperature:

$$\rho_R = \frac{\pi^2}{30} g_{*,i} T^4, \tag{7}$$

where $g_{*,i}$ is the effective number of degrees of freedom for species i. For a single neutrino species,

$$g_{*,\nu} = 2\frac{7}{8}\left(\frac{4}{11}\right)^{4/3} = 0.45, \tag{8}$$

The energy density fraction for a single neutrino species, Ω_ν is

$$\Omega_\nu = \frac{\rho_\nu}{3H_0^2/8\pi G} = 5.6 \text{ x } 10^{-6}, \tag{9}$$

using the current temperature of $T = 2.725$ K. Calculating a limit on the number of extra neutrino degrees yields a limit on relativistic energy density from particles other than photons and neutrinos, above which any CGWB could not exceed. If the number of extra neutrino degrees of freedom is N, then

$$\Omega_{GW} = 5.6N \text{ x } 10^{-6}. \tag{10}$$

Smith et al perform calculations to constrain the energy density of the CGWB using observational data that limits the parameters. Though not stated in [1], the calculation likely proceeds by use of a package like COSMOMC that takes a set of parameters and the constraints from observations (via CMB data, LSS data, Lyman-alpha forests) to calculate the likelihood that a particular unknown parameter will fall within a range and still yield the overall observational results. Here, the set of parameters is the non-relativistic matter density, $\Omega_m h^2$, the baryon density $\Omega_b h^2$, the scalar spectral index n_s, the power-spectrum amplitude A_s, the optical depth τ to the surface of last scatter, and the angle θ subtended by the first acoustic peak. Likelihoods are typically calculated by finding a likelihood function

$$L \propto e^{-\frac{1}{2}\chi^2}, \tag{11}$$

where

$$\chi^2 = \sum_j \left(\frac{\tilde{y_j} - y_j}{\Delta \tilde{y_j}} \right)^2 , \tag{12}$$

where $\tilde{y_j}$ is the observed value, $\Delta\tilde{y_j}$ is the uncertainty in the observed value, and y_j is the theoretical predicted value. When $\chi = 0$, the gaussian likelihood function has a maximum, and this is the best fit to the observations. When looking at the likelihood plots for different parameters, the peak is the average, best fit value of the parameter to meet the observations, and the acceptable range for the parameter is defined as being within a number of standard deviations away from the average.

The data sets used separately or in combination are those of the CMB power spectra, galaxies (LSS), and the Lyman-alpha forest. The LSS data comes from Sloan Digital Sky Survey and refers to the distribution of matter in the universe on large scales. The Lyman-alpha forest is another measure of large-scale structure coming from spectral lines observed in the light from quasars. The spectra lines correspond to those of excited hydrogen and are used to infer how much hydrogen and in what regions is it located between the Earth and the quasar.

A. Adiabatic

Smith et al calculate the constraints on the CGWB energy density for two scenarios: whether the CGWB has adiabatic or non-adiabatic initial conditions. If a gravitational wave background exists under adiabatic conditions (so that the energy density of all relativistic primordial particle species was the same), and since the GWs correspond to massless spin-2 particles, the CGWB would behave like the gas of massless neutrinos and the extra neutrino degrees of freedom constrained by the CMB data yields a maximum limit on the energy density of the CGWB. The results for the likelihood for $\Omega_{GW}h^2$ are shown on the left-hand side of Figure 2. Use of only the CMB data gives the strongest bound, and addition of the galaxy and Lyman-alpha tests to the CMB data shifts the bound out to slightly higher (factor of 2) $\Omega_{GW}h^2$.

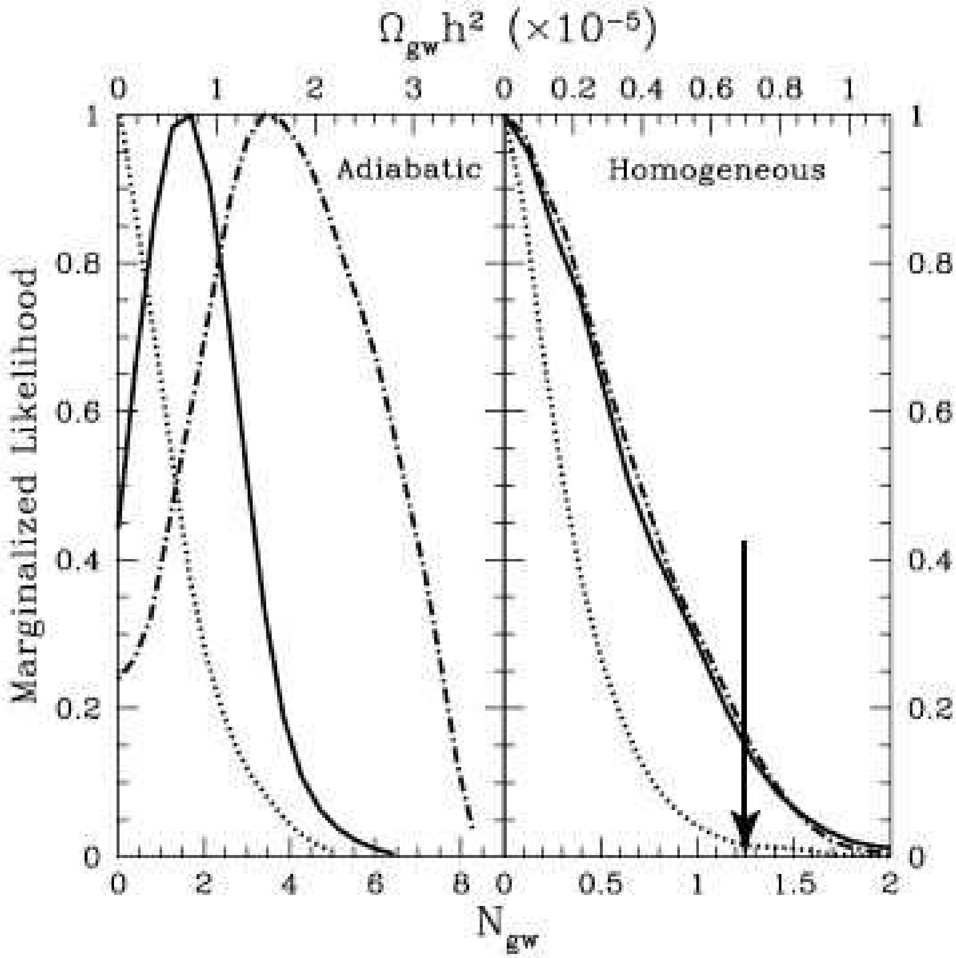

FIG. 2: Marginalized likelihoods for the CGWB energy density. Dotted curve uses only CMB data, solid curve uses CMB+galaxies+Lyman, and the dot-dash curve uses CMB+galaxies+Lyman if neutrino masses are allowed to be non-zero and marginalized over. Figure taken from [1]. The left side corresponds to the adiabatic calculation, the right side to the non-adiabatic. The arrow on the non-adiabatic side is showing 95% confidence level for the CMB+galaxies+Lyman+variable neutrino mass.

B. Non-adiabatic

If inflation is responsible for the CGWB by producing quantum fluctuations in the metric, then the CGWB would be non-adiabatic. Being a tensor perturbation, the CGWB perturbations are directly proportional to the quantum fluctuations, but the scalar perturbations of the particle species energy densities are related to the inflaton field in a more complicated way. However, the perturbations to the particles are adiabatic - all the particle species receive the same density perturbations. In this case, the CGWB will not have the same energy density perturbations as the other particle species. Further details on this case are in preparation by the authors of [1], but the results of the likelihood analysis for homogeneous/non-adiabatic initial conditions for the CGWB are shown in Figure 2. For the completely combined data set, CMB+galaxies+Lyman+variable neutrino masses, an upper limit on $\Omega_{GW}h^2 \leq 6.9$ x 10^{-6} is obtained. This limit is with 95% confidence, which means that 95% or 2 standard deviations fall under this value. Again, the bound is about 2-fold stronger if only the CMB data is used in the analysis.

V. DISCUSSION

The homogeneous/non-adiabatic case yields stronger limits on the CGWB density than the adiabatic case, and though the limit is near the BBN limit, it extends to smaller frequency, over the range that previously have no prior constraints. The true lower bound on the range of frequencies over which the above constraints are valid was not completely determined, but argued that if the wavelength of the gravitational wave is larger than the horizon at recombination, then the gravitational waves would not be propagating as massless modes but would be frozen outside the horizon. In an inflation model, perturbation modes have a nearly scale invariant spectrum, and most of the modes start inside the horizon, as the inflaton field is slowly varying, and cross out sometime before the end of inflation and beginning of the radiation-dominated universe, freezing out at that point. Then as the universe continues to expand, the modes eventually come back into the hubble radius (the hubble radius being on the order of the horizon size) and propagate. The assumptions above that the gravitational waves could be analyzed similarly to the relativistic massless species, the neutrino, requires that they are propagating modes. For this reason, they conservatively cut off the range two decades before this scale is reached, at $f \approx 10^{-15}$ Hz. The limits calculated from these two cases are shown relative to the other bounds discussed in Figure 3. Also noted is the fact that according to the inflation described above, which leaves the gravitational wave spectrum almost scale-invariant, the number of modes crossing back into the horizon is a function of time. The progressively crossed out in the primordial era, and then came back inside during the matter or radiation dominated eras [9]. In this case, the energy

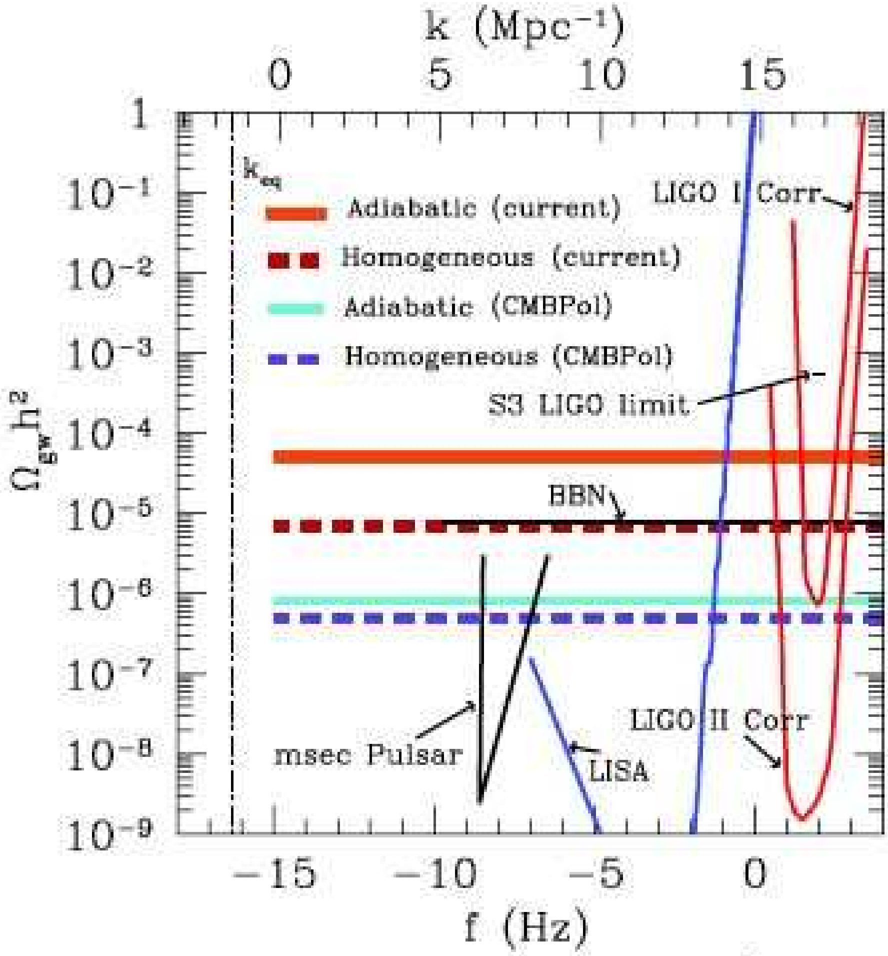

FIG. 2: The gravitational-wave density $\Omega_{gw}h^2$ versus frequency. The BBN constraint corresponds to a limit of 1.4 extra neutrino degrees of freedom. We also show our constraints, from current CMB, galaxy, and Lyman-α data, for a CGWB with adiabatic primordial perturbations ("Adiabatic (current)") and for homogeneous initial conditions ("Homogeneous (current)"), as well as our forecasts for the sensitivities if current CMB data are replaced by data from CMBPol. Also shown are the reaches of LIGO and LISA. BBO (not shown) should go deeper, but primarily at frequencies ~ 1 Hz. Large-angle CMB fluctuations (also not shown) constrain $\Omega_{gw}h^2 \lesssim 10^{-14}$, but only at frequencies $\lesssim 10^{-16}$ Hz. The LIGO S3 upper limit is from Ref. [18] and the msec pulsar curve is from Refs. [5, 19].

FIG. 3: gravitational wave density vs. frequency. Figure taken from [1].

density would change not as simply as for a relativistic particle species, $\rho_R \propto a^{-4}$, but only presents a logarithmic correction to the work here. However, these constraints are not really applicable to inflationary models anyways, because they have already been constrained at very low frequencies by the CMB to have energy densities $\Omega_{GW} h^2 \leq 10^{-15}$ [10].

to CGWB that is generated by some process that decouples the CGWB from the particles species energy density, however,

VI. CONCLUSIONS

A new limit to the energy density of a cosmological gravitational wave background has been given for a frequency range that has been yet unexplored. The limit is essentially the same as that given by BBN, $\Omega_{GW} h^2 \leq 10^{-6}$, but extends 5 orders of magnitude to lower frequencies. Though a calculation corresponding to non-adiabatic initial conditions of the CGWB was carried out, it does not really correspond to inflationary mechanisms which produce scale-invariant GW spectrum and have been constrained, though not over the same frequency range as the one here, to much lower amplitude. However, this limit would apply to gravitational waves created by some other mechanisms that lived in this frequency range.

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