Critical analysis of "Vacuum energy: myths and reality" by Grigory Volovik

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Contents

1 The two cosmological constant problems

The supernova observations [1] show that our Universe cannot be explained by a model containing only matter and radiation. If we assume the missing component is a cosmological constant and the Universe is flat, the observations give a fit to its density parameter $\Omega_{\Lambda 0} \sim 0.7$ corresponding to an energy density of $\rho_{\Lambda(fitted)} \sim (10^{-3}\ eV)^4$. The main candidate for cosmological constant is the energy of the vacuum.

The vacuum energy contribution of a bosonic field of mass m is the sum of the zero-point energies of its modes

$$\rho_{\Lambda\ bos} = \int_0^\infty \frac{4\pi k^2 dk}{(2\pi)^3} \frac{1}{2}\sqrt{k^2+m^2} \tag{1}$$

which is infinite. The leading contribution comes from high momenta $k \gg m$. Imposing an ultraviolet cutoff at Planck scale ($k_{max} = M_{planck}$) leads to

$$\rho_{\Lambda\ bos} \sim \int^{M_{pl}} k^3\ dk \sim M_{pl}^4 \sim (10^{27}\ eV)^4. \tag{2}$$

The vacuum energy of fermions is similar but negative. The vacuum energies of bosons and fermions do not cancel exactly due to different degrees of freedom and the total vacuum energy will be of the same order so we obtain the notorious discrepancy between theory and experiment

$$\frac{\rho_\Lambda(theory)}{\rho_\Lambda(fitted)} \sim \left(\frac{10^{27}\ eV}{10^{-3}\ eV}\right)^4 = 10^{120}, \tag{3}$$

which is the first cosmological constant problem. We could get away by saying the vacuum does not gravitate. Then the second problem is to explain why the measured value of $\rho_{\Lambda 0}$ is not exactly zero but of the order of the matter density.

The 120 orders of magnitude discrepancy could be alleviated by supersymmetry (SUSY) that matches fermionic with bosonic species. It is believed SUSY is spontaneously broken at energies below $M_{SUSY} \sim 10^3\ GeV = 10^{12}\ eV$. Above that energy, the positive contributions of bosons exactly cancel out the negative contributions of fermions in the total vacuum energy. The cancelation does not occur below M_{SUSY} so the final vacuum energy is

$$\rho_{\Lambda(SUSY)} \sim \int^{M_{SUSY}} k^3\ dk \sim M_{SUSY}^4 \sim (10^{12}\ eV)^4 \tag{4}$$

and the corresponding discrepancy is

$$\frac{\rho_\Lambda(SUSY)}{\rho_\Lambda(fitted)} \sim (10^{15})^4 = 10^{60}. \tag{5}$$

2 Vacuum of the Universe paralleled to ground state in quatum liquids

G. Volovik's ideas are presented somehow sketchy in [2]. The main idea is to parallel the vacuum of the Universe to the well studied ground states of condensed matter quantum liquids (liquid ^4He or ^3He). The instant payoff in the condensed matter case is that we know the underlying microscopic theory of the ground state (many-body Schroedinger equation for the atoms of the liquid) and we can calculate the ground state energy. On the other hand, a low-energy observer living in the liquid sees only the low energy excitations of the ground state (quasiparticles, phonons) and does not have access to the high energy constants necessary to regularize the seemingly divergent vacuum energy.

In the case of the Universe, we do not know the underlying Planckian theory of the vacuum structure. We do not know if the vacuum consists of discrete elements (analogues to the atoms of the quantum liquid) or not. All we have is an effective theory (the Standard Model) of the excitations about the vacuum which we call particles (analogue of the quasiparticles in the condensed matter case). Correspondingly, the naive expectation that we can calculate the vacuum energy from the effective theory is too optimistics since probably we do not have access to the necessary regularization constants and terms. There does not exist a proof that the Planck energy scale has something to do with the vacuum energy and should be used as a regulator like in (2).

3 Bogolyubov theory of weakly interacting Bose gas

In this section we will outline the calculation of the ground state energy of a weakly interacting Bose gas which is the simplest tractable case. In the next section we will compare the exact result with the naive (and wrong) predictions of a low-energy observer. We are following mainly [3] and some bits of [4, 5, 6].

3.1 Bosonic Fock space

Consider the one-particle momentum eigenfunctions (plane waves)

$$\phi_{\mathbf{k}}(\mathbf{x}) = \langle \mathbf{x} | \mathbf{k} \rangle = \frac{1}{\sqrt{V}} e^{i\mathbf{k}\cdot\mathbf{x}} \tag{6}$$

normalized in a rectangular box of volume $V = L_x\, L_y\, L_z$. Periodic boundary conditions are imposed leading to allowed values of momentum

$$\mathbf{k} = 2\pi \left(\frac{n_x}{L_x}, \frac{n_y}{L_y}, \frac{n_z}{L_z} \right), \quad n_{xyz} = 0, \pm 1, \pm 2, \ldots \tag{7}$$

These functions span the one-particle Hilbert space and are orthonormal

$$\int \mathbf{d^3x}\; \phi^*_{\mathbf{k}}(\mathbf{x})\; \phi_{\mathbf{k'}}(\mathbf{x}) = \delta_{\mathbf{k}\,\mathbf{k'}}. \tag{8}$$

A two-particle bosonic Hilbert space cant be constructed by using a basis of fully symmetrized products $S[\phi_{\mathbf{k_1}}(\mathbf{x_1})\phi_{\mathbf{k_2}}(\mathbf{x_2})] \propto \phi_{\mathbf{k_1}}(\mathbf{x_1})\phi_{\mathbf{k_2}}(\mathbf{x_2}) + \phi_{\mathbf{k_2}}(\mathbf{x_1})\phi_{\mathbf{k_1}}(\mathbf{x_2})$ for all possible combinations of $(\mathbf{k_1}, \mathbf{k_2})$.

We can continue constructing bosonic N-particle Hilbert spaces for $N =$ 0, 1, 2, 3 . . . The direct sum of all such spaces is called Fock space. All basis vectors in the Fock space are uniquely specified by the occupation numbers

$|n_{k_1},\ n_{k_2}, \ldots n_k \ldots\rangle$ where n_k shows how many times $\phi_{\mathbf{k}}$ appears in each term of the symmetrized basis vector.

We can define creation operators $a_{\mathbf{k}}$ for all allowed $\mathbf{k's}$ that lead from an N-particle subspace to N+1-particle subspace:

$$a_k^+ \, |\ldots n_k \ldots\rangle = \sqrt{n_k + 1}\ \ |\ldots n_k + 1 \ldots\rangle \qquad (9)$$

and the Hermitian conjugate annihilation operators that lead from N into N-1-particle subspace:

$$a_k \, |\ldots n_k \ldots\rangle = \sqrt{n_k}\ \ |\ldots n_k - 1 \ldots\rangle\,. \qquad (10)$$

N is determined by the initial vector $|\ldots n_k \ldots\rangle$, $N = n_{k_1} + n_{k_2} + \ldots + n_k + \ldots$ The defined operators satisfy the usual commutators for bosons

$$[a_k, a_{k'}] = 0, \ \ [a_k^+, a_{k'}^+] = 0, \ \ [a_k, a_{k'}^+] = \delta_{k\,k'}. \qquad (11)$$

The convenience of a_k and a_k^+ is that other operators can be expressed in terms of them.

3.2 One- and two-particle operators in second quantized form

Consider an operator that acts in the N-particle subspace of Fock space and is a symmetric sum of one-particle operators

$$T = t(1) + t(2) + \ldots + t(N) = \sum_{\alpha=1}^{N} t(\alpha), \qquad (12)$$

e. g. $t(\alpha) = \mathbf{p}_\alpha{}^2/2m$ is the kinetic energy of α-th particle and T is the total kinetic energy of the system. Each t_α acts only on states $|\mathbf{k}\rangle_\alpha$ of particle α and can be written as

$$t(\alpha) = \sum_{k\,k'} \langle k|_\alpha \, t(\alpha) \, |k'\rangle_\alpha \, |k\rangle_\alpha \, \langle k'|_\alpha\,. \qquad (13)$$

The matrix elements $\langle k|_\alpha \, t(\alpha) \, |k'\rangle_\alpha = t_{k\,k'}$ are independent of α since $t(\alpha)$ are operators of the same functional form. Consequently, the total operator is

$$T = \sum_{k\,k'} t_{k\,k'} \sum_{\alpha=1}^{N} |k\rangle_\alpha \, \langle k'|_\alpha\,. \qquad (14)$$

The operator $\sum_{\alpha=1}^{N} |k\rangle_\alpha \langle k'|_\alpha$ acting on an N-particle state, substitutes $|k'\rangle$ with $|k\rangle$ state everywhere does not matter which particle is in $|k'\rangle$. It can be written as

$$\sum_{\alpha=1}^{N} |k\rangle_\alpha \langle k'|_\alpha = a_k^+ \, a_{k'} \;\; \text{(acting on N-particle state)}. \tag{15}$$

The above is very plausable since $a_{k'}$ annihilates $|k'\rangle$ and a_k^+ creates a $|k\rangle$ state at its place. The total operator is

$$T = \sum_{k\,k'} t_{k\,k'} \; a_k^+ \; a_{k'}. \tag{16}$$

For reasons we are not going to explain, the expression of an operator in terms of creation/annihilation operators is called a second quantized form.

In the case of kinetic energy operators:

$$t(\alpha) = \frac{-\hbar^2 \nabla_\alpha^2}{2m} \tag{17}$$

$$t_{k\,k'} = \langle k| \, \frac{-\hbar^2 \nabla_\alpha^2}{2m} \, |k'\rangle = \int \mathbf{d^3x} \; \phi_{\mathbf{k}}^*(\mathbf{x}) \frac{\hbar^2 k'^2}{2m} \; \phi_{\mathbf{k'}}(\mathbf{x}) = \frac{\hbar^2 k^2}{2m} \delta_{k\,k'} \tag{18}$$

$$T = \sum_{\alpha=1}^{N} \frac{-\hbar^2 \nabla_\alpha^2}{2m} = \frac{\hbar^2}{2m} \sum_k k^2 \, a_k^+ \, a_k \;\; \text{(acting on N-particle state)}. \tag{19}$$

Similarly to one-particle operators, we can consider a symmetric sum of two-particle operators acting in an N-particle subspace

$$F = \frac{1}{2} \sum_{\substack{\alpha,\beta=1 \\ \alpha \neq \beta}}^{N} f^{(2)}(\alpha, \beta), \tag{20}$$

where for example $f^{(2)}(\alpha, \beta)$ could describe interaction between particles α and β. The sum in F is over all possible combinations of particles within the N particles and can be written in second quantized form as

$$F = \frac{1}{2} \sum_{k_1\, k_2\, k_3\, k_4} \langle k_1 \; k_2| \, f^{(2)} \, |k_3 \; k_4\rangle \, a_{k_1}^+ \, a_{k_2}^+ \, a_{k_3} \, a_{k_4}. \tag{21}$$

If $f^{(2)}(\alpha,\beta) = U(\mathbf{x}_\alpha - \mathbf{x}_\beta)$, is a two-particle interaction potential that depends only on the relative coordinates of the particles, the matrix element is

$$\langle k_1\ k_2|\, U(\mathbf{x}_\alpha - \mathbf{x}_\beta)\, |k_3\ k_4\rangle = \tag{22}$$

$$\frac{1}{V^2}\int d^3x_\alpha d^3x_\beta e^{-ik_1x_\alpha} e^{-ik_2x_\beta} U(x_\alpha - x_\beta) e^{-ik_3x_\alpha} e^{-ik_4x_\beta} = \tag{23}$$

$$\frac{1}{V^2}\int d^3x_\alpha d^3x_\beta e^{-i(k_1x_\alpha + k_2x_\beta)} \frac{1}{V}\sum_q U_q e^{iq(x_\alpha - x_\beta)} e^{-i(k_3x_\alpha + k_4x_\beta)} = \tag{24}$$

$$\frac{1}{V^3}\sum_q U_q \int d^3x_\alpha e^{-ix_\alpha(q-k_1+k_3)} \int d^3x_\beta e^{-ix_\beta(q+k_2-k_4)} = \tag{25}$$

$$= \frac{1}{V}\sum_q U_q\ \delta_{q-k_1+k_3,\,0}\ \delta_{q+k_2-k_4,\,0}, \tag{26}$$

where $U_\mathbf{q} = \int \mathbf{d^3x}\ e^{-\mathbf{q}\cdot\mathbf{x}}\ U(\mathbf{x})$ is the Fourier transform of U. Finally, we can write the second quantized form of F:

$$F = \frac{1}{2}\sum_{k_1\,k_2\,k_3\,k_4} \langle k_1\ k_2|\, U(\mathbf{x}_\alpha - \mathbf{x}_\beta)\, |k_3\ k_4\rangle\, a^+_{k_1}\, a^+_{k_2}\, a_{k_3}\, a_{k_4} = \tag{27}$$

$$= \frac{1}{2V}\sum_{q\,k_3\,k_4} U_q\ a^+_{k_3+q}\, a^+_{k_4-q}\, a_{k_3}\, a_{k_4}. \tag{28}$$

3.3 Approximation of the Hamiltonian in the vicinity of the ground state

The general Hamiltonian of a gas of N bosons interacting by a two-particle interaction potential is

$$H = \frac{-\hbar^2}{2m}\sum_{\alpha=1}^{N}\nabla^2_\alpha + \frac{1}{2}\sum_{\substack{\alpha,\,\beta=1\\ \alpha\neq\beta}}^{N} U(\mathbf{x}_\alpha - \mathbf{x}_\beta). \tag{29}$$

It acts in the N-particle subspace of the Fock space. The momentum N-particle eigenstates $|n_{k_1},\ n_{k_2}, \ldots n_k \ldots\rangle$ are eigenstates of the above Hamiltonian only if the interaction potential is zero but we can still use them as a basis to span Fock space. The whole machinery developed in the previous section can be used now to write H in second quantized form

$$H = \frac{1}{2m}\sum_k k^2\, a^+_k\, a_k + \frac{1}{2V}\sum_{q\,k_3\,k_4} U_q\ a^+_{k_3+q}\, a^+_{k_4-q}\, a_{k_3}\, a_{k_4}\ (\hbar = 1 \text{ units}). \tag{30}$$

The above is perfectly general. Now we specialize to the case of weakly interacting bosons. The ground state of a gas of non-interacting bosons is all of them condensing in the $k = 0$ single-particle state. Analogously, we expect the ground state of the weakly interacting boson gas to have a significant number of the particles, N_0, in the $k = 0$ single-particle state and a small number of particles, $N' = N - N_0 \ll N$ in $k \neq 0$ states. We want to diagonalize the Hamiltonian in the vicinity of the ground state hence we neglect terms in it corresponding to interactions between excited particles (of order N'^2) and only consider interactions of excited particles ($k \neq 0$) with the condensate ($k = 0$) (of order $N'N$) or interactions of condensate particles among themselves (of order N_0^2):

$$H \approx \frac{1}{2m}\sum_k k^2\, a_k^+\, a_k + \frac{U_0}{2V} a_0^+ a_0^+ a_0 a_0 + \frac{1}{V}\sum_k{}' (U_0 + U_k) a_0^+ a_0 a_k^+ a_k \tag{31}$$

$$+\frac{1}{2V}\sum_k{}' U_k (a_k^+ a_{-k}^+ a_0 a_0 + a_0^+ a_0^+ a_k a_{-k}), \tag{32}$$

where the primed sums are over all allowed $\mathbf{k}'s$ except $\mathbf{k} = \mathbf{0}$. The effect of a_0^+ and a_0 on the ground state is

$$a_0^+\, |N_0, \ldots\rangle = \sqrt{N_0 + 1}\ \, |N_0 + 1, \ldots\rangle \tag{33}$$

$$a_0\, |N_0, \ldots\rangle = \sqrt{N_0}\ \, |N_0 - 1, \ldots\rangle \tag{34}$$

Since $N_0 \sim N \sim 10^{23}$ is a huge number, we can approximate $\sqrt{N_0 \pm 1} \approx \sqrt{N_0}$ and consider the states $|N_0, \ldots\rangle$, $|N_0 \pm 1, \ldots\rangle$ as physically the same. Moreover, the commutator $[a_0^+, a_0] = 1 \ll N_0$. All that allows us to approximate these operators as c-numbers:

$$a_0^+ = a_0 = \sqrt{N_0}. \tag{35}$$

The Hamiltonian becomes

$$H \approx \frac{1}{2m}\sum_k{}' k^2\, a_k^+\, a_k + \frac{U_0\, N_0^2}{2V} + \tag{36}$$

$$+\frac{N_0}{V}\sum_k{}' [(U_0 + U_k) a_k^+ a_k + \frac{1}{2} U_k (a_k^+ a_{-k}^+ + a_k a_{-k})]. \tag{37}$$

The number of particles in the $\mathbf{k} = \mathbf{0}$ one-particle state is

$$N_0 = N - \sum_k{}' a_k^+ a_k. \tag{38}$$

Substituting that in H gives:

$$H \approx \frac{1}{2m} \sum_k{}' k^2 \, a_k^+ \, a_k + \frac{U_0 \, N^2}{2V} + \tag{39}$$

$$+\frac{N}{V} \sum_k{}' U_k a_k^+ a_k + \frac{N}{2V} U_k (a_k^+ a_{-k}^+ + a_k a_{-k}) + \text{ terms with 4 a's.} \tag{40}$$

Following Bogolyubov, we neglect the anharmonic terms with 4 a operators since they are of order $(N'/V)^2 \ll (N/V)^2$. What is left of the Hamiltonian is quadratic in a's and can always be diagonalized by performing a Bogolyubov transformation to new creationg/annihilation operators α_k^+, α_k:

$$a_k = u_k \, \alpha_k + v_k \, \alpha_{-k}^+ \tag{41}$$

$$a_k^+ = u_k \, \alpha_k^+ + v_k \, \alpha_{-k} \tag{42}$$

Requiring the usual bosonic commutators $[\alpha_k, \alpha_{k'}] = 0, \;\; [\alpha_k^+, \alpha_{k'}^+] = 0, \;\; [\alpha_k, \alpha_{k'}^+] = \delta_{k\,k'}$ and demanding disapperance of the off-diagonal terms in the Hamiltonian leads after some algebra to

$$H \approx \frac{U_0 \, N^2}{2V} + \frac{1}{2} \sum_k{}' (\omega_k - \frac{k^2}{2m} - n \, U_k) + \sum_k{}' \omega_k \, \alpha_k^+ \, \alpha_k, \tag{43}$$

where $n = N/V$ is the particle density and

$$\omega_k = \sqrt{(\frac{k^2}{2m} + n \, U_k)^2 - (n \, U_k)^2} \tag{44}$$

is the energy of the α_k^+ excitations which are called quasiparticles.

3.4 Particles and quasiparticles, ground and vacuum states.

An important clarification is in order. The original particles (^4He atoms) are created by the a_k^+ operators acting on their vacuum state $|0, \, 0, \, \ldots\rangle$ which contains no particles at all. The ground state of the N particles is $|N_0, \ldots\rangle$ and contains N_0 particles in $k = 0$ one-particle state and a small fraction, $N' = N - N_0$, in excited $k \neq 0$ one-particle states. The α_k^+ operators create excited states (quasiparticles) from the ground state. The ground state $|N_0, \ldots\rangle$ contains no quasiparticles, hence it plays the role of a vacuum state for the quasiparticles:

$$\alpha_k \, |N_0, \ldots\rangle = 0, \; \forall \, \mathbf{k}. \tag{45}$$

3.5 Quaziparticle energies, speed of sound, ground state energy and pressure

For small momenta, $k \ll \sqrt{mnU_k}$, the energy of the quasiparticles given in (44) is

$$\omega_k \approx k\sqrt{\frac{nU_0}{m}}, \tag{46}$$

where we approximated U_k with U_0 for small k. The corresponding low-energy quasiparticles are phonons propagating at the speed of sound

$$c = \sqrt{\frac{nU_0}{m}}. \tag{47}$$

For big momenta, $k \gg \sqrt{mnU_k}$, expanding the quasiparticle energy in (44) to second order we get

$$\omega_k \approx \frac{k^2}{2m} + nU_k - \frac{mn^2U_k^2}{k^2}. \tag{48}$$

The ground state energy is the part of the Hamiltonian in (43) without α_k operators

$$E_0 = \frac{U_0 \, N^2}{2V} + \frac{1}{2}{\sum_k}'(\omega_k - \frac{k^2}{2m} - n \, U_k). \tag{49}$$

We have to define the interaction potential in order to compute it. For a contact type potential $U(\mathbf{x}_\alpha - \mathbf{x}_\beta) = U \, \delta(\mathbf{x}_\alpha - \mathbf{x}_\beta)$, where U is a constant giving the potential strength, the Fourier transform is $U_k = U = const$. For big momenta

$$\omega_k - \frac{k^2}{2m} - nU_k \approx -\frac{mn^2U_k^2}{k^2} = -\frac{mn^2U^2}{k^2}. \tag{50}$$

That means that the sum $\sum_k{}'$ in the ground state energy goes like $-\int \frac{dk}{k^2} \sim -ln(k)$ and the ground state energy is ultraviolet divergent (logarithmically). To remedy that, we substitute the contact potential with a short-range potential having the same s-wave scattering lenght [6]. The results will be the same as long as U is contained only in the scattering amplitude. The difference is that the short-range potential can be treated by perturbation theory and the contact potential cannot as we saw above. The final outcome of that regularization is that the ground state energy aquires an extra term which cancels out the divergence [5]:

$$E_0 = \frac{U\,N^2}{2V} + \frac{1}{2}\sum_k{}'(\omega_k - \frac{k^2}{2m} - n\,U + \frac{mn^2U^2}{k^2}). \tag{51}$$

The first term in E_0 represents the interactions of the atoms in the $k = 0$ condensate among themselves. The otherwise divergent sum over the zero point energy of phonons, $\frac{1}{2}\sum_k{}'\omega_k$, is balanced by three counter terms that render if finite. Substituting the sum with an integral gives

$$E_0 = \frac{U\,N^2}{2V} + \tag{52}$$

$$+\frac{1}{2}\int_0^\infty \frac{V}{(2\pi)^3}4\pi k^2 dk \left[\sqrt{(\frac{k^2}{2m} + n\,U_k)^2 - (n\,U_k)^2} - \frac{k^2}{2m} - n\,U + \frac{mn^2U^2}{k^2}\right] = \tag{53}$$

$$= \frac{U\,N^2}{2V} + \frac{V}{4\pi^2}m^{3/2}\frac{32}{15}(nU)^{5/2} = \tag{54}$$

$$= V\left(\frac{1}{2}Un^2 + \frac{8}{15\pi^2}m^{3/2}U^{5/2}n^{5/2}\right), \tag{55}$$

which is the result quoted in [7]. The pressure of the ground state (which is the quasiparticle vacuum) is

$$P = -\frac{\partial E_0}{\partial V} == \frac{1}{2}Un^2 + \frac{4}{5\pi^2}m^{3/2}U^{5/2}n^{5/2}, \tag{56}$$

which differs from the one obtained in [7] mainly by the fact that both terms are positive. We have 'weakly interacting Bose gas' when the second terms in E_0 and P are much smaller than the first terms.

4 Low-energy observer living in weakly interacting Bose gas

A low-energy observer living in the gas of weakly interacting bosons will notice the quasiparticle excitations of energies $\omega_k \approx ck$. She will construct an effective theory by quantizing the field equations and will calculate for the energy of the quasiparticle vacuum $\frac{1}{2}\sum c\,k$ which of course is infinite. At low energy she has access only to the speed of sound of the quasiparticles $c = \sqrt{nU/m}$ but not to the separate constants n, U, or m that describe the underlying structure of the 'vacuum'. Lacking this information, she will not be able to regularize properly the infinite sum. She will totally miss the dominant term in the vacuum energy (the first term in (51)) which comes from the 'vacuum' constituents interaction.

5 Low-energy observer in the Universe

Similarly to the observer in the Bose gas, we are low-energy observers of the Universe. Our effective theory of excitations about the vacuum is the Standard Model. Trying to calculate the vacuum energy, we get of course an infinite sum. Since we know nothing of the underlying structure of the vacuum (if there is such a thing) we are not able to regularize it properly. Using the Planck mass as an upper limit cutoff is simply put naive. Nothing guarantees that constant has something to do with the vacuum structure or with the energy spectrum of the zero-point modes. The analogy between ground states in quantum liquids and the Standard Model vacuum is of course possible if the latter has constituent 'atoms' like the atoms of liquid ^4He. That may not be necessarily the case.

6 Droplet of real liquid ^{4}He at zero temperature

The Bogolyubov theory of weakly interacting Bose gas is nice because it is tracktable analytically. In reality, the interactions between the atoms of liquid ^4He are not weak and much more sophisticated and evolved methods of computation have to be applied. According to Volovik though [2, 7], the ground state energy should satisfy the general thermodynamic equality of Gibbs-Duhem

$$E - \mu N = TS - PV. \tag{57}$$

If we consider a large spherical droplet of ^4He at zero gravity and assume that the droplet radius is big enough so we can neglect the pressure from surface tension ($\sim 2\sigma/radius$), then the pressure inside the droplet is $P = 0$. That is very distinct from the weakly interacting Bose gas. The liquid state can exist without external pressure while the gas needs a positive external pressure (see (56)).

At zero temperature, $T = 0$, the droplet is in the ground state of energy E_0:

$$E_0 - \mu N = -PV = 0. \tag{58}$$

Volovik argues [2, 7] that the 'relevant ground state energy' is

$$E_{vac} = E_0 - \mu N = 0, \tag{59}$$

not E_0 itself. The argument is that in order to find the ground state, one has to minimize $\langle\psi| \hat{H} |\psi\rangle$ keeping the number of particles (^4He atoms), $\langle\psi| \hat{N} |\psi\rangle$, constant. Equivalently, one could minimize $\langle\psi| \hat{H} - \mu\hat{N} |\psi\rangle$, where the chemical potential μ serves as a Lagrange multiplier enforcing the particle conservation. This is like considering the system at fixed chemical potential and variable number of particles istead of fixed number of particles and a chemical potential that depends on it. Both methods give the same result at the end, but it is more convenient to work with fixed chemical potential and to go to fixed number of particles in the final results [4, 6].

[My comment: The fact that it is calculationally more convenient to minimize $\hat{H} - \mu\hat{N}$ instead of $\hat{H}$ does not prove that 'the relevant ground energy' is $E_0 - \mu N$ instead of E_0.]

Volovik gives another argument that $E_0 - \mu N$ is invariant under shift in the definition of particle energy while E_0 is not. If, for example, the energy of each particle is increased by a constant by including its rest energy, then $E_0 \to E_0 + N \times const$ and $\mu \to \mu + const$ leaving $E_0 - \mu N$ invariant. The next implication is that it is the quantity $E_0 - \mu N$ that gravitates and it is nice that it does not depend on the choice of zero.

[My comment: Independence of such choice is not required. The Einstein assumption that all forms of energy gravitate actually does not allow for arbitrary choice of zero energy. For example, the absolute energy of a photon is measurable by absorbing it and is $h\nu$. According to Einstein, that energy corresponds to a gravitational mass of $h\nu/c^2$. Other energies can be measured with respect to the photon energy, for example the energy of a particle at rest can be measured by annihilation with its anti-particle and measuring the energies of the resulting photons. That shows, there exist an operational definition of energy (tells you how to measure it) which does not allow for arbitrary choice of zero. Measuring of the absolute energy of the vacuum will require it conversion to photons somehow. That does not sound crazier than the idea that vacuum has constituents.]

7 Droplet of liquid ^{4}He at $T \neq 0$

If the liquid ^4He droplet is at nonzero temperature, it will have a gas of phonon excitations on top of the vacuum. Using again the Gibbs-Duhem relation (57) and the Volovik's definition of vacuum energy we get the equation of state of the vacuum component (still assumed zero temperature for the vacuum)

$$\rho_{vac} = E_{vac}/V = (E - \mu N)/V = -P_{vac}, \tag{60}$$

where ρ_{vac} is the vacuum energy density. The phonon gas is analogous to a photon gas only the speed of light must be replaced by the speed of sound. Correspondingly, the usual equation of state holds

$$P_{phon} = \frac{1}{3}\rho_{phon}, \tag{61}$$

where ρ_{phon} does not include the zero-point energy of the phonons (that energy is simply dropped in the derivation of the Planck's law of black body radiation). That is why the vacuum must be added as a separate component. The total pressure of the self-sustained drop is zero as before:

$$0 = P_{vac} + P_{phon} = -\rho_{vac} + \frac{1}{3}\rho_{phon}. \tag{62}$$

Volovik concludes that while the vacuum energy of the droplet at zero temperature was zero (see (59)), when the vacuum is 'disturbed' by the presence of the phonons at non-zero temperature, the new vacuum energy is of the order of the disturbing phonon energy

$$\rho_{vac} = \frac{1}{3}\rho_{phon} \tag{63}$$

in order to render the total pressure zero again. He points out that the vacuum energy density is governed by the macroscopic requirement of zero total pressure but not by the specific quantum structure of the vacuum under consideration.

8 Equation of state of the Universe vacuum?

Volovik applies the same Gibbs-Duhem relation to the vacuum of the Universe [2, 7, 8]

$$E_0 - \mu N = (TS - P_{vac}V)_{T=0} = -P_{vac}V. \tag{64}$$

Arguing that the 'relevant vacuum energy' is $E_{vac} = E_0 - \mu N$ he gets

$$\rho_{vac} = E_{vac}/V = -P_{vac}, \tag{65}$$

which is exactly the equation of state of a cosmological constant.

9 Nonzero cosmological constant: Einstein static and de Sitter Universes

Volovik transfers the ideas from the liquid ^4He droplet to the Universe [8]. He assumes that the vacuum is always described by a cosmological constant: $\rho_{vac} = -P_{vac}$. Then he reasons that the Universe is a self-contained system non-interacting with its environment (whatever that means). Similarly to the Helium droplet, the total pressure of the Universe must be zero. That statement is in disagreement with the standard cosmology based on the Einstein equations but he gets away with it by reinterpreting the Einstein tensor as effective gravitational energy density and pressure.

The Einstein equations for a universe containing a cosmological constant (vacuum) and matter fields (with some equation of state parameter ω^{matter}) are

$$\frac{G_{\mu\nu}}{8\pi G} = \frac{\Lambda g_{\mu\nu}}{8\pi G} + T_{\mu\nu}^{matter}, \tag{66}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\mu}$ is the Einstein tensor, Λ is the cosmological constant, and $T_{\mu\nu}^{matter}$ is the energy-momentum tensor of the matter fields (radiation included). Moving everything to the right side we get

$$0 = T_{\mu\nu}^{grav} + T_{\mu\nu}^{vac} + T_{\mu\nu}^{matter}, \tag{67}$$

$$T_{\mu\nu}^{grav} = -\frac{G_{\mu\nu}}{8\pi G}, \quad T_{\mu\nu}^{vac} = \frac{\Lambda g_{\mu\nu}}{8\pi G}. \tag{68}$$

In the case of Robertson-Walker metric, $T_{\mu\nu}^{grav}$ really has the form of energy-momentum tensor of ideal fluid

$$T_{\mu\nu}^{grav} = \rho^{grav} U_\mu U_\nu + P^{grav}(U_\mu U_\nu - g_{\mu\nu}), \tag{69}$$

$$\rho^{grav} = \frac{-3}{8\pi G}\left(H^2 + \frac{k}{a^2}\right), \quad P^{grav} = \frac{1}{8\pi G}\left(2\frac{\ddot{a}}{a} + H^2 + \frac{k}{a^2}\right). \tag{70}$$

In that interpretation the Einstein equations are simply

$$\begin{aligned} 0 &= \rho^{grav} + \rho^{vac} + \rho^{matter} \\ 0 &= P^{grav} + P^{vac} + P^{matter}, \end{aligned} \tag{71}$$

which state that the total pressure and energy density of the universe are zero. The total pressure is analogous to the total pressure of the liquid ^4He droplet. The equations of state of the different components are

$$\begin{aligned} P^{grav} &= -\frac{\left(2\frac{\ddot{a}}{a} + H^2 + \frac{k}{a^2}\right)}{3\left(H^2 + \frac{k}{a^2}\right)}\rho^{grav}, \quad P^{vac} = -\rho^{vac} \\ P^{matter} &= \omega^{matter}\rho^{matter}. \end{aligned} \tag{72}$$

In the Einstein static universe ($a = 1 = const$) containing cosmological constant and matter we have $P^{grav} = -\rho^{grav}/3$. Using (71) and (72) we get the relation

$$\rho^{vac} = \frac{1}{2}\rho^{matter}(1 + 3\omega^{matter}). \tag{73}$$

Consider now a flat ($k = 0$) de Sitter universe containing only cosmological constant. From (71) we get $P^{grav} = -P^{vac} = \rho^{vac} = -\rho^{grav}$. Using the equation of state for gravity in (72), we get the differential equation

$$\frac{\ddot{a}}{a} = H^2 = \frac{\dot{a}^2}{a^2} \tag{74}$$

with solution $a(t) = A\, e^{Ht}$, where A and H are constants. From (70) we get

$$\rho^{vac} = -\rho^{grav} = \frac{3H^2}{8\pi G} = const. \tag{75}$$

In both universes the vacuum energy density is not zero but of the same order as the 'perturbations' which are the matter fields in the Einstein static universe, and the universe expansion H in the flat de Sitter universe.

10 Vacuum energy tracking the perturbations?

Volovik tries to extend the idea that the vacuum energy is of the same order as the perturbations (zero in the absence of perturbations) in any type of universe. That would resolve the first and the second cosmological constant problems. It works for the universes considered in the previous section because they require a **constant** cosmological constant and of course it has to be proportional to the other constant parameters. Unfortunately, if one insists on the equation of state, $\rho_{vac} = -P_{vac}$, Einstein equations forbid time evolution of such component. That is seen by taking a covariant derivative of the Einstein equations in (66):

$$\frac{1}{8\pi G}\nabla^\mu G_{\mu\nu} = \frac{1}{8\pi G}\partial_\nu \Lambda + \nabla^\mu T^{matter}_{\mu\nu}. \tag{76}$$

According to the purely geometrical Bianci identities $\nabla^\mu G_{\mu\nu} = 0$. Assuming the energy of the matter fields is separately conserved $\nabla^\mu T^{matter}_{\mu\nu} = 0$, we obtain $\partial_\nu \Lambda = 0$ i.e. Λ is a constant in space and time. In order to allow for the cosmological constant to track the other component densities, Volovik modifies the Einstein equations adding 'relaxation terms'. The interested reader is refered to [8].

11 Summary and discussion

The ideas of Grigory Volovik relevant to this paper can be found in his book [9] which is a compendium of many of his articles. In summary, they are:

1. The vacuum of the Universe is similar to the ground state of quantum liquids. That suggests a low energy observer having only an effective theory about that vacuum may not be able to regularize the vacuum energy properly.

2. The 'relevant ground state energy' that gravitates is $E_0 - \mu N$ not E_0. I found that idea has no proper justification and is equivalent to a pure guess.

3. The vacuum state of any system obeys the Gibbs-Duhem thermodynamic relation. That and the previous idea leads to the universal equation of state for the vacuum of the quantum liquid or the Universe: $\rho_{vac} = -P_{vac}$, which is exactly the equation of state of the cosmological constant in General Relativity.

4. The vacuum energy density is zero in the absence of perturbations (no gravity, no matter fields, zero temperature, no Universe expansion). In the presence of perturbations it is of the same order of magnitude as the perturbations. That works in universes that require a constant cosmological constant like Eintein's static universe or the flat de Sitter universe. In the case of arbitrary universe, Einstein equations do not allow time variation of the cosmological constant (the vacuum according to Volovik) and have to be modified. That idea 'explains' why the cosmological constant is small but nonzero by simply postulating it.

5. The total pressure in the Universe is zero since it is a self-sustaining system that does not interract with its environment. That works if the Einstein tensor is reinterpreted as effective energy density and pressure of gravity. That is an interesting idea but it is equivalent to the Einstein equations and leads to no new results.

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