Three year WMAP results: implications for inflation

Xibo Zhang
*Department of Physics, University of Chicago, Chicago, IL 60637, USA**
(Dated: June 2, 2006)

The three year WMAP results give the scalar spectral index (at $k = 0.002/\text{Mpc}$)$n_s = 0.951 + 0.015/-0.019$ (Table 5[1]). This limits the range of parameters for various inflationary models. In this paper, we calculate, for a general form of inflationary potential, (1) the evolution of the unperturbed universe (2) several properties of the density fluctuation, including n_s and r (the ratio of tensor to scalar power spectrum) under the linear perturbation condition. Simulations are done for the $\lambda\phi^4$ model. The calculations and simulations are consistent within 6%. The calculated results are then compared with the three year WMAP data. The data disfavors the $\lambda\phi^4$ model with minimal gravitational coupling.

PACS numbers:

I. INTRODUCTION

The standard Big-Bang cosmology has several puzzles: flatness, horizon, monopole , homogeneity and isotropy (but not exactly so), Inflation is invented[2] to solve these puzzles with two simple assumptions:

- The early universe is dominated by dark-energy, which is characterized by some field("the inflaton").
- At some stage, the dark energy decays into matter and radiation.

Guth's model is improved after his work. In many present models a slow-roll period is assumed to exist. Linear perturbation calculations are carried out to find the information of the fluctuations. The fluctuations are responsible for the anisotropy of the cosmic microwave background (CMB), which is measured by the Wilkinson Microwave Anisotropy Probe (WMAP) and other experiments. The three year WMAP results give the cosmological parameters with unprecedented accuracy, including the scalar spectral index $n_S = 0.951 + 0.015/-0.019$. This puts a number of inflationary model into test. In this paper, we start from an unperturbed universe under the slow-roll assumption, calculate its evolution process; we then do the first order perturbation, get the expressions of relevant quantities, including the scalar spectral index; with these theoretical results we perform a simulation to trace the evolution of the inflaton field and the scale factor at the same time, test the slow-roll picture, visualize the behavior of density fluctuation spectrum at a large wavenumber(k) range, and numerically get the scalar spectral index n_S. Our simulation results agree with the previous calculations within 6%. With the calculated scalar spectra index and ratio of tensor to scalar energy spectrum, we compare them with the three year WMAP result. The data disfavors the minimal gravitational coupling model. Two methods are discussed for putting constraints on inflationary models.

II. THE UNPERTURBED UNIVERSE

In a flat homogeneous and isotropic Robertson-Walker universe, the Friedmann equations are

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \tag{1}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{2}$$

where a is the scale factor, ρ the energy density, and p the pressure; we use $\hbar = c = 1$, and $G = 1/M_P^2$ is the gravitational constant. In the inflationary models, the early universe is dominated by a scalar inflaton field ϕ, with

*Electronic address: xibo@uchicago.edu

the Lagrangian density

$$\hat{\mathcal{L}} = -\frac{1}{2}\left(\nabla_\mu\phi\right)^2 - V(\phi) = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2a^2}\sum_{i=1}^{3}(\partial_i\phi)^2 - V(\phi) \tag{3}$$

and the equation of motion (E.O.M.)

$$\ddot{\phi} - \frac{1}{a^2}\nabla^2\phi + 3H\dot{\phi} + V'(\phi) = 0 \tag{4}$$

where $H = \dot{a}/a$ is the Hubble parameter. In the unperturbed case $\partial_i\phi = 0$.

The energy density and pressure can be calculated by the energy-momentum tensor, and are given by

$$\rho = \rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{5}$$

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{6}$$

Thus the Friedmann equation and the E.O.M. are rewritten as

$$H^2 = \frac{8\pi}{3M_P^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \tag{7}$$

$$\frac{\ddot{a}}{a} = -\frac{8\pi}{3M_P^2}\left(\dot{\phi}^2 - V(\phi)\right) \tag{8}$$

$$0 = \ddot{\phi} + 3H\dot{\phi} + V'(\phi) \tag{9}$$

A. The slow-roll approximation

Define two slow-roll parameters

$$\epsilon = \frac{1}{16\pi}M_P^2\left(\frac{V'}{V}\right)^2 \tag{10}$$

$$\eta = \frac{1}{8\pi}M_P^2\frac{V''}{V} \tag{11}$$

We require[3]

$$\epsilon << 1 \tag{12}$$

$$|\eta| << 1 \tag{13}$$

as well as

$$\dot{\phi}^2 << V(\phi) \tag{14}$$

and

$$\dot{\phi} \approx -\frac{V'(\phi)}{3H} \tag{15}$$

which is equivalent to $|\ddot{\phi}| << |V'(\phi)|$.

Under the above three assumptions, the field evolves as

$$\phi = \phi_0 - \frac{V'(\phi)}{3H}t \tag{16}$$

and the scale factor evolves as

$$a(t) = \frac{8\pi}{M_P^2}\int_{\phi(t)}^{\phi_0}\frac{V}{V'}\mathrm{d}\phi \tag{17}$$

The number of e-fold is

$$N = \ln\left(\frac{a_f}{a_i}\right) = \frac{8\pi}{M_P^2}\int_{\phi_f}^{\phi_i}\frac{V}{V'}\mathrm{d}\phi \tag{18}$$

The validity of these approximations can be verified by numerical simulations.

III. LINEAR PERTURBATION

A. Relate density perturbation to spatial curvature perturbation

Introducing the perturbation for quantity f:

$$\delta f(\mathrm{x}, t) = f(\mathrm{x}, t) - \bar{f}(t) \tag{19}$$

the density contrast δ and its Fourier transformation δ_k is defined as

$$\delta \equiv \frac{\delta\rho(\mathrm{x}, t)}{\rho(\mathrm{x}, t)} \tag{20}$$

$$\overset{\text{To 1st order}}{=} \frac{\delta\rho(\mathrm{x}, t)}{\bar{\rho}(t)} \tag{21}$$

and

$$\delta(\mathrm{x}, t) = \sum_\mathrm{k}\delta_\mathrm{k}(t)e^{i\mathrm{k}\cdot\mathrm{x}} \tag{22}$$

Liddle and Lyth derive the relation between δ_k and $\mathcal{R}_\mathrm{k}$ (equation(2.30))[4]

$$\left(\frac{aH}{k}\right)^2\delta_\mathrm{k} = \frac{2+2w}{5+3w}\mathcal{R}_\mathrm{k} \tag{23}$$

where $k = |\mathrm{k}|$, $w = p/\rho$, and $\mathcal{R}_\mathrm{k}$ is the Fourier transformation of $\mathcal{R}$, a measure of the perturbation in the spatial curvature of co-moving hyper-surfaces. To facilitate comparisons, a quantity $\delta_H(\mathrm{k})$ is defined as the density constrast at $k = aH$ for each k.

1. *A puzzle*

We are NOT clear whether $k = aH$ is for the horizon exit or horizon entry. To get Liddle and Lyth's final result (5.18)[4], we need to assume

$$w = 0 \tag{24}$$

at the point of interest, which seem to be in the matter dominated period. Thus

$$\delta_H(\mathrm{k}) = \frac{2}{5}\mathcal{R}_\mathrm{k} \tag{25}$$

B. Relate spatial curvature perturbation to filed perturbation

Liddle and Lyth derive the relation between the spatial curvature perturbation $\mathcal{R}_\mathrm{k}$ and the field perturbation ϕ_k(equation (A.11))[4]

$$\mathcal{R} = \frac{H}{\dot{\phi}}\delta\phi \tag{26}$$

To the first order $H\delta\phi/\dot{\phi} \approx \bar{H}\delta\phi/\dot{\bar{\phi}}$, and we get

$$\mathcal{R}_\mathrm{k} = \frac{\bar{H}}{\dot{\bar{\phi}}}\delta\phi_\mathrm{k} \tag{27}$$

C. The power spectrum

The discrete and continuous Fourier transformations of a real function f are

$$\begin{aligned} f(\mathrm{x},t) &= \sum_{\mathrm{k}} f_{\mathrm{k}}(t)e^{i\mathrm{k}\cdot\mathrm{x}} \qquad (28)\\ &= L^3\sum_{\mathrm{k}}\left(\frac{2\pi}{L}\right)^3\frac{1}{(2\pi)^3}f_{\mathrm{k}}(t)e^{i\mathrm{k}\cdot\mathrm{x}}\\ &\equiv L^3\int\frac{\mathrm{d}^3\mathrm{k}}{(2\pi)^3}f_{\mathrm{k}}e^{i\mathrm{k}\cdot\mathrm{x}}\\ &\equiv \int\frac{\mathrm{d}^3\mathrm{k}}{(2\pi)^3}F_{\mathrm{k}}e^{i\mathrm{k}\cdot\mathrm{x}} \qquad (29)\\ F_{\mathrm{k}} &\equiv L^3 f_{\mathrm{k}} \qquad (30)\end{aligned}$$

where k takes discrete values. For an isotropic perturbation, f_{k} only depends on $|\mathrm{k}|$.

The spectrum is defined as[4]

$$\begin{aligned} P_f(k) &\equiv \left(\frac{L}{2\pi}\right)^3 4\pi k^2 k|f_{\mathrm{k}}|^2 \qquad (31)\\ &= \frac{1}{(2\pi)^3L^3}4\pi k^2 k|F_{\mathrm{k}}|^2 \qquad (32)\end{aligned}$$

where L is the size of the cubic box in which the field is defined. It satisfies

$$\begin{aligned} &\int_0^\infty \mathrm{d}k\frac{P_f(k)}{k}\\ &= \frac{1}{L^3}\int\frac{\mathrm{d}^3\mathrm{k}}{(2\pi)^3}|F_{\mathrm{k}}|^2\\ &= \frac{1}{L^3}\int\mathrm{d}^3\mathrm{x}\int\frac{\mathrm{d}^3\mathrm{k}}{(2\pi)^3}F_{\mathrm{k}}e^{i\mathrm{k}\cdot\mathrm{x}}\int\frac{\mathrm{d}^3\mathrm{k}'}{(2\pi)^3}F^*_{\mathrm{k}'}e^{-i\mathrm{k}'\cdot\mathrm{x}}\\ &= \frac{1}{L^3}\int\mathrm{d}^3\mathrm{x}f(\mathrm{x},t)^2\\ &= \left\langle f(\mathrm{x},t)^2\right\rangle \qquad (33)\end{aligned}$$

1. *Another obscure point*

For the quantum field, the Fourier component f_{k} is replaced by the expectation value of the operator $\hat{f}_{\mathrm{k}}$ under the corresponding quantum state. Liddle and Lyth calculate the vacuum expectation value of $\delta\phi_{\mathrm{k}}$ as (equation (5.31))[4]

$$\begin{aligned} <|\delta\phi_{\mathrm{k}}|^2> &= \frac{H^2}{2k^3} \qquad (34)\\ &\approx \frac{\bar{H}^2}{2k^3} \qquad (35)\end{aligned}$$

D. Power spectrum of δ_H

Putting (25),(27) and (34) together, we get

$$\begin{aligned}
P_{\delta_H} &= \left[L^3 \frac{1}{(2\pi)^3} 4\pi k^2 k < |\delta_H(\mathrm{k})|^2 >\right]_{k=aH} \\
&= \left[L^3 \frac{k^3}{2\pi^2} \frac{4}{25} < |\mathcal{R}_\mathrm{k}|^2 >\right]_{k=aH} \\
&= \left[L^3 \frac{k^3}{2\pi^2} \frac{4}{25} \left(\frac{\bar{H}}{\dot{\bar{\phi}}}\right)^2 < |\delta\phi_\mathrm{k}|^2 >\right]_{k=aH} \\
&= \left[L^3 \frac{k^3}{2\pi^2} \frac{4}{25} \left(\frac{\bar{H}}{\dot{\bar{\phi}}}\right)^2 \frac{\bar{H}^2}{2k^3}\right]_{k=aH} \\
&= \left[L^3 \frac{\bar{H}^4}{25\pi^2 \dot{\bar{\phi}}^2}\right]_{k=aH} \\
&= \left[L^3 \frac{9\bar{H}^6}{25\pi^2 V'(\phi)^2}\right]_{k=aH} \\
&= \left[L^3 \frac{512\pi V(\phi)^3}{75 V'(\phi)^2}\right]_{k=aH}
\end{aligned} \tag{36}$$

E. Scalar spectral index n_s

The scalar spectral index is defined as

$$n_s(k) \equiv 1 + \left[\frac{\mathrm{d}\ln(P_{\delta_H})}{\mathrm{d}\ln(k)}\right]_{k=aH=\dot{a}} \tag{37}$$

Thus it is calculated as follows:

$$\begin{aligned}
n_s(k) - 1 &= \left[k \frac{\mathrm{d}}{\mathrm{d}k} \ln(P_{\delta_H})\right]_{k=aH=\dot{a}} \\
&= \left[k \frac{\mathrm{d}\phi}{\mathrm{d}k} \frac{\mathrm{d}}{\mathrm{d}\phi} \ln(P_{\delta_H})\right]_{k=aH=\dot{a}} \\
&= \left[\left(k\frac{\mathrm{d}\phi}{\mathrm{d}k}\right)\left(3\frac{V'}{V} - 2\frac{V''}{V'}\right)\right]_{k=aH=\dot{a}} \\
&= \left[\left(\dot{a}\frac{\mathrm{d}\phi}{\mathrm{d}\dot{a}}\right)\left(3\frac{V'}{V} - 2\frac{V''}{V'}\right)\right]_{k=aH=\dot{a}} \\
&= \left[\left(\frac{\dot{a}}{a}\frac{\dot{\phi}}{\ddot{a}/a}\right)\left(3\frac{V'}{V} - 2\frac{V''}{V'}\right)\right]_{k=aH=\dot{a}} \\
&= \frac{-1}{8\pi} M_P^2 \left[3\left(\frac{V'}{V}\right)^2 - 2\frac{V''}{V}\right]_{k=aH} \\
&= [-6\epsilon + 2\eta]_{k=aH}
\end{aligned} \tag{38}$$

In specific cases, the spectral index n_S can be expressed in terms of the number of e-folds N and the potential parameters. For example, for $V \propto \phi^\alpha$, the spectral index is[1, 4]

$$n_S - 1 = -\frac{2+\alpha}{2N} \tag{39}$$

F. Ratio of tensor to scalar power spectrum (at $k = 0.002$/Mpc)

The ratio of tensor to scalar power spectrum is defined as (see equation(4.29)[4])

$$r = \frac{\Sigma_l^2(\text{grav})}{\Sigma_l^2(\text{density})} \quad (40)$$

where Σ_l^2 is the variance of the distribution of the multipole a_l^m of the cosmic microwave background (CMB) anisotropy. For the l which is much larger than 1 and well below the cutoff, Liddle and Lyth calculate r as[4]

$$r = 12.4\epsilon \quad (41)$$

Later Peiris et al.[5] and Spergel et al.[1] use

$$r = 16\epsilon \quad (42)$$

In the following calculation we will use the latter one. Specifically, for $V(\phi) \propto \phi^\alpha$ with the number of e-folds N, we have $r = \frac{4\alpha}{N}$.[1]

IV. A EXAMPLE: $\lambda\phi^4$ INFLATON FIELD

A. The unperturbed case

Suppose

$$V = \lambda\phi^4(\lambda > 0) \quad (43)$$

The slow-roll parameters are

$$\epsilon = \frac{1}{\pi}\frac{M_P^2}{\phi^2} \quad (44)$$

$$\eta = \frac{3}{2\pi}\frac{M_P^2}{\phi^2} \quad (45)$$

We choose $\phi_f = M_P$ at the end of inflation.

If we want the number of e-fold to be $N = 60$, i.e.

$$60 = \frac{\pi\phi_i^2}{M_P^2} - \pi \quad (46)$$

the initial field is $\phi_i = M_P\sqrt{1 + 60/\pi} \approx 4.48M_P$. The field decays with time exponentially when $\phi > M_P$:

$$\phi = \phi_i \exp\left[-\sqrt{\frac{2\lambda}{3\pi}}M_P t\right] \quad (47)$$

The time duration of inflation is

$$T = \sqrt{\frac{3\pi}{2\lambda}}\frac{1}{M_P}\ln\left(\frac{\phi_i}{M_P}\right) \approx \frac{3.26}{\sqrt{\lambda}}\frac{1}{M_P} \quad (48)$$

These results are tested by numerical simulations. Using $\phi_i = 4.48M_P$, $\lambda = 1$, we get the evolution of field, as shown in figure (1). The evolution is well approximated by an exponential decay when $\phi > 0.3M_P$:

$$\phi/M_P = 4.50\exp[-\frac{tM_P}{2.18}]$$

where the amplitude and time constant agrees well with $\phi_0 = 4.48$ and $\sqrt{3\pi/(2\lambda)} = 2.17$. Besides, the number of e-fold is 63 with $\phi_f = 1$, which has a 5% discrepancy.

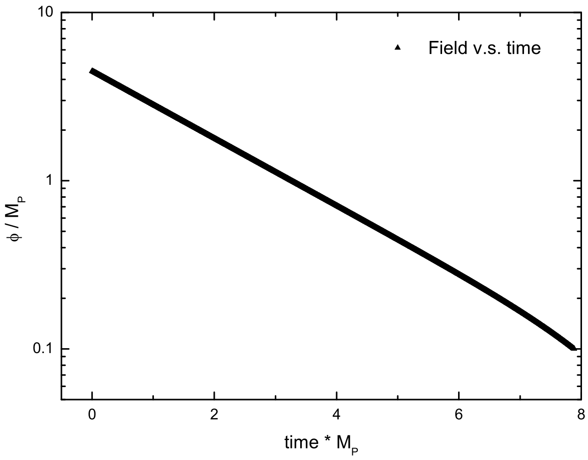

FIG. 1: The simulated evolution of $\lambda\phi^4$ field, with $lambda = 1$, $\phi_i = 4.48M_P$.

B. Linear perturbation: scalar spectral index

For the $\lambda\phi^4$ inflaton field,

$$\ln P_{\delta_H} \;=\; \left[\ln \phi^6\right]_{k=aH} + \text{constant} \tag{49}$$

Thus

$$\begin{aligned} n_s(k) \;&=\; 6\left[\frac{\mathrm{d}(\phi/M_P)}{\mathrm{d}\ln k/k_i}\right]_{k=aH} \\ &=\; \frac{\mathrm{d}(\phi/M_P)}{\mathrm{d}\ln(aH/(a_iH_i))} \end{aligned} \tag{50}$$

The ϕ/M_P v.s. $\ln(aH/(a_iH_i))$ dependence is shown in figure (2); it is actually the $\ln(P_{\delta_H})$ v.s. $(\ln(k/k_i))_{k=aH}$ dependence up to a constant added.

We fit the small k part (shown in figure (3)). The slope is -0.053, which corresponds to $n_S - 1$ ($N \approx 60$), and agrees with (39) within 6% discrepancy.

From figure (2) (3), as well as the enlarged figure (4) we make several observations:

- n_S is NOT constant for the $\lambda\phi^4$ model, at least in the slow-roll scheme. There is NO reason one should believe the power spectrum is scale invariant.
- For small k, n_S is approximately constant and red-tilted; the fitted slope is consistent with theoretical calculation within 6%. We don't know the error of the theoretical calculation, thus not comparing the discrepancy with error.
- For large $k(\ln(k/k_i) > 58.4$, i.e., $k > 2.2 \times 10^{25}k_i$ in figure (2)), the power spectrum is multi-valued, hence n_S is multi-valued. At $\phi = 0.466M_P$, aH begins to decrease – although a is still increasing; the two values for the same k might correspond to 'horizon entry' and 'horizon exit'(not necessarily the first time!)

C. Ratio of tensor to scalar power spectrum; comparison with three year WMAP results

We get

$$r \;=\; \frac{4\alpha}{N} = \frac{4\times 4}{60} = \frac{4}{15} = 0.267 \tag{51}$$

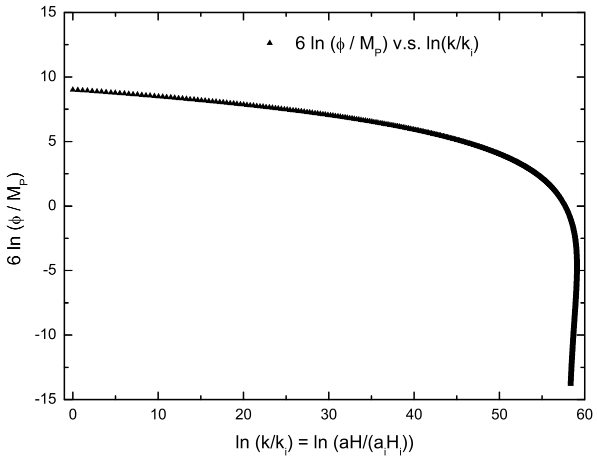

FIG. 2: The simulated dependence of $\ln(P_{\delta_H})$ v.s. $(\ln(k/k_i))_{k=aH}$, up to a constant added; parameters are the same.

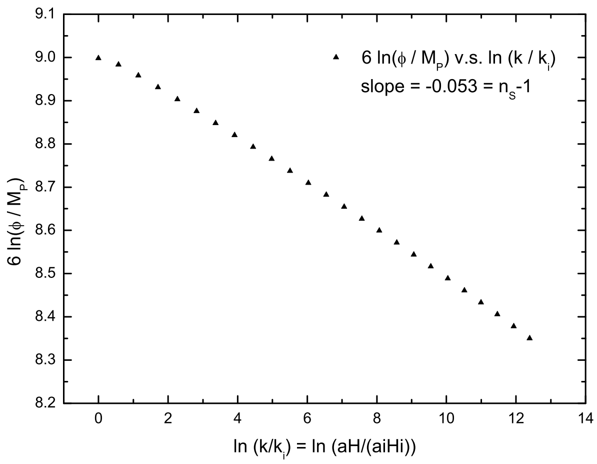

FIG. 3: The small-wavenumber part of the $\ln(P_{\delta_H})$ v.s. $(\ln(k/k_i))_{k=aH}$ curve.

Hence the $\lambda\phi^4$ potential requires $n_S = 0.947$, $r = 0.267$. Compared with the three year WMAP results, the prediction just lies at the outer edge of the 3σ region.[1] This means, even if the data does not rule out the $\lambda\phi^4$ theory, it disfavors the theory. To save the theory, one might need to use more complicated mechanism beyond the minimal gravitational coupling.[1]

D. Idea to put constraints on inflationary models

Comparison between the calculated n_S and r and the three year WMAP data puts constraints on various models. There are two ways to do the calculation:

- The main step in the calculation is to find out the field value when $k = aH$. This idea is carried out by de Vega and Sanchez[**?**].

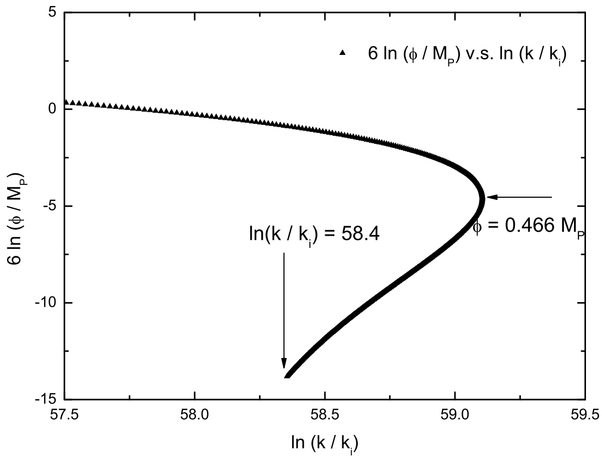

FIG. 4: The multi-value part of the $\ln(P_{\delta_H})$ v.s. $(\ln(k/k_i))_{k=aH}$ curve.

- In the above calculation we follow another line:(1) numerically trace the evolutions of both the field and the scale factor at the same time – get a one-to-one correspondence between the field and aH; (2)use this correspondence to study the power spectrum (obtained from the field) as a function of $\ln(k)$(the wavenumber $k = aH$).

V. CONCLUSION

Both theoretical calculations and numerical simulations are done for (1) the unperturbed universe (2) linear perturbation for density fluctuation. The density fluctuation power spectrum is plotted in a large wavenumber range, and fitted in the small wavenumber range. The scalar spectral index is both calculated and obtained from simulation. The calculated and simulated results agree within 6%. The calculated results are compared with the three year WMAP data; the data disfavors the $\lambda\phi^4$ model with minimal gravitational coupling. Further constraints on inflationary models are discussed.

Acknowledgments

The author wishes to acknowledge Prof. S. Carroll for inspiring discussions and encouragement.

[1] D. N. Spergel et al., astro-ph/0603449.
[2] A. H. Guth, Phys. Rev. D **23**, 347 (1981).
[3] D. H. Lyth and A. Riotto, Phys. Rep. **314**, 1 (1999).
[4] A. R. Liddle and D. H. Lyth, Phys. Rep. **231**, 1 (1993).
[5] H. V. Peiris et al., ApJS **148**, 213 (2003).