Magnetic field generation from cosmological perturbations

Priyanka Jindal

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1 Introduction

Most astrophysical systems have been observed to possess magnetic fields of varying strengths – from 1 gauss in interstellar media, to 10^{-6} gauss at the galactic scale. The origin of these primordial fields is not well- understood.

This report discusses the paper by Takahashi, et. al. (2005) [1], where the authors present a theoretical calculation to show how magnetic fields can arise from cosmological perturbations. To achieve the same, they consider the interaction of a plasma of electrons, protons, and photons. This approach is different from the previous attempts. Some previous attempts required the existence of a seed magnetic field. Typically, a finite magnetic field present in a system would decay with time, owing to the finite conductivity of the medium. To explain the steady state of astrophysical magnetic fields, a dynamo mechanism is proposed, that sustains the growth of the seed magnetic field constantly. But attempts to generate seed fields, like the Biermann mechanism [2], that have been applied to explain fields in various large-scale structures, supernovae remnants, and protogalaxies, have had limited success. This is because of the narrow range of magnitude the theory produces ($\sim 10^{-21} - 10^{-16}$ Gauss), and also because of the constraint that the fields thus obtained have characteristic length scales. Other attempts to generate seed fields use different physical mechanisms. It has also been proposed that large scale magnetic fields could have originated from primordial fields generated during the inflationary era of the early universe.

In [1] the authors consider the coupling between electrons, protons and the photon plasma, after recombination. Each component is treated as a perfect fluid. The electrons and protons interact via Coulomb forces. The photons interact with both the electrons and the protons, but the (Thomson) scattering cross-section for the proton-photon process is suppressed by a factor of $(m_e/m_p)^2$, and thus can be neglected to second order in perturbation. This preferential coupling introduces an asymmetry in the motion of the electrons and protons, and is the cause of an electric current, which in turn generates a magnetic field.

2 Discussion

The main components of the plasma in the radiation era (after neutrino decoupling, and before recombination) consists primarily of photons, free electrons and protons. The three exist in thermal equilibrium and can be treated as perfect fluids. Thus the covariant Euler equations for the electrons and protons can be written as follows.

$$(\delta^i_\mu + u^i u_\mu)(T_p^{\mu\nu}{}_{;\nu} + T_{\gamma p}^{\mu\nu}{}_{;\nu}) = C_{pe}^{(C)i} + C_{p\gamma}^{(T)i} \tag{1}$$

$$(\delta^i_\mu + u^i u_\mu)(T_e^{\mu\nu}{}_{;\nu} + T_{\gamma e}^{\mu\nu}{}_{;\nu}) = C_{ep}^{(C)i} + C_{e\gamma}^{(T)i} \tag{2}$$

Here $T^{\mu\nu}_{p(e)}$ are the energy-momentum stress tensors for the proton (electron) fluids, and $T^{\mu\nu}_{\gamma p(e)}$ are the stress tensors for the photon-proton (electron) interaction. The subscript "$;\nu$" denotes the ν-th component of the covariant derivative of the tensor. u and δ are the four-velocity and Kronecker delta respectively. Also note that indices $\mu, \nu = 0, 1, 2, 3$ and $i = 1, 2, 3$. The $C^{(C)i}$ terms represent the i-th component of the force due to momentum exchange in collisions between the electrons and protons due to Coulomb interaction. The $C^{(T)i}_{p(e)\gamma}$ terms denote the i-th component of the collisional force due to Thomson scattering between the photons and protons(electrons). Note that in the absence of the collision terms, energy and momentum of the fluids is conserved (the energy-momentum tensors are divergenceless).

Also, the collision terms here are the first moments of the collision factors one obtains in the Boltzmann transport equation. In other words, for a species y with a distribution function $f(p_i)$, interaction with species z, through a process X

$$C_{yz}^{(X)i}[f(p_i] = \int \frac{d^3p}{2\pi^3} p^i C_{yz}^{(X)}[f(p_i)] \tag{3}$$

This is not the most convenient notation, but we keep it, to conform with [1]. To simplify the equations above, we can use the general form of the energy-momentum tensor for a fluid with energy density ρ and pressure p

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu} \tag{4}$$

and the energy-momentum tensor for electromagnetic fields

$$T_{\gamma}^{\mu\nu} = \left(F^{\mu\sigma}F_{\sigma}{}^{\nu} + \frac{1}{4}\delta^{\mu\nu}(F^{\rho\sigma}F_{\rho\sigma})\right) \tag{5}$$

This simplies equations (1) and (2) as follows

$$(\delta^i_\mu + u^i u_\mu)T^{\mu\nu}{}_{;\nu} = (\rho + p)u^\mu u^i{}_{;\mu} + (\delta^i_\mu + u^i u_\mu)P^{,\mu} \tag{6}$$

and

$$(\delta^i_\mu + u^i u_\mu)T_{\gamma}^{\mu\nu}{}_{;\nu} = -j^\nu F^\mu{}_\nu \tag{7}$$

Here we used the fact that $u^\mu u_\mu = -1$ and $\partial_\nu T^\nu{}_\mu = -j_\mu$. P denotes pressure, j_μ denotes the electric current, and $F^{\mu\nu}$ is the electromagnetic field tensor.

To simplify the right hand side we note that $C_{pe}^{(C)i} = -C_{ep}^{(C)i}$ causes the magnetic field to diffuse. In the highly conducting plasma of the early universe any magnetic fields would disappear rapidly through the $e-p$ Coulomb interaction, and would not be able to sustain itself. Thus we can ignore the Coulomb scattering collision terms. Also, the collision cross-section between the proton and photon is much weaker than that between the electron and photon, as mentioned earlier (suppressed by a factor of $(m_e/m_p)^2$). Thus we can also neglect the proton-photon Thomson scattering term in the original equations, and consider the effect of the electron-photon Thomson scattering on the equations of motion alone.

Now consider the process

$$\gamma(p_i) + e^-(q_i) \longrightarrow \gamma(p_i') + e^-(q_i') \tag{8}$$

The collision integral $C_{\gamma e}^{(T)}[f(p_i]$ for this process is calculated next. Here $f(p_i)$ is the photon distribution function, and $f(q_i)$ is the electron distribution function, which is taken to be the equilibrium Maxwellian because the perturbations due to scattering with photons are weak. Thus

$$f(x,\mathbf{q}) = (2\pi)^3 x_e n_e (2\pi m_e T_e)^{-3/2} exp\Big(\frac{-(q-m_e v_e)^2}{2m_e T_e}\Big) \tag{9}$$

where x_e is the ionization fraction and n_e is the number density of the electrons. T_e is the electron effective temperature and v_e is the mean electron velocity. Also,

$$Dq = \frac{d^3q}{2\pi^3 2E_e(q)} \tag{10}$$

is the Lorentz invariant phase space element. Thus the form of the collision integral is

$$\begin{aligned} C^{(T)}_{\gamma e}[f(p_i] = & \frac{2\pi^4}{p}\int Dp' \int Dq \int Dq' \\ & \times |M|^2 \delta[E_\gamma(p) + E_e(q) - E_\gamma(p') - E_e(q')]\delta^{(3)}(p_i + q_i - p'_i - q'_i) \\ & \times \{f_\gamma(p'_i)f_e(q'_i) - f_\gamma(p_i)f_e(q_i)\} \end{aligned} \tag{11}$$

Here $|M|^2$ is the Thomson scattering amplitude. Using the expansion

$$\delta(p+q-p'-q') = \delta(p-p') + \Big(\frac{1}{m_e p}(\mathbf{p}-\mathbf{p}').\mathbf{q} + \frac{1}{m_e p}(\mathbf{p}-\mathbf{p}')^2\Big)p\frac{\partial\delta(p-p')}{\partial p'} + h.o. \tag{12}$$

we get

$$C^{(T)}_{\gamma e}[f(p_i] \sim \frac{\pi n_e}{4m_e^2 p}\int \frac{d^3p'}{2\pi^3 p'}|M|^2\{f_\gamma(p'_i) - f_\gamma(p_i)\}\Big(\delta(p-p') + (p_i - p'_i)u^i_e\frac{\partial\delta(p-p')}{\partial p'}\Big) \tag{13}$$

Thus the collision term in the Euler equation (2) is given as

$$\begin{aligned} C^{(T)i}_{\gamma e}[f(p_i] = & \int \frac{d^3p}{2\pi^3}p^i C^{(T)}_{\gamma e}[f(p_i)] \\ & = \frac{4\sigma_T \rho_\gamma n_e}{3}\Big[(u^i_e - u^i_\gamma) + \frac{1}{8}u_{ej}\Pi^{ij}_\gamma\Big] \end{aligned} \tag{14}$$

Here σ_T is the Thomson scattering cross-section, and Π_γ^{ij} is the photon anisotropic stress tensor observed in the CMB spectrum, which causes the deviation of the universe from being isotropic and homogeneous (refer [3]). The successive moments of the photon and electron distribution function used here are:

$$\int \frac{d^3p}{(2\pi)^3} p f_\gamma(p_i) = \rho_\gamma \qquad \text{(photon energy density)} \tag{15}$$

$$\int \frac{d^3p}{(2\pi)^3} p^i f_\gamma(p_i) = \frac{4}{3}\rho_\gamma u_\gamma^i \tag{16}$$

$$\int \frac{d^3p}{(2\pi)^3} p^i f_e(p_i) = \rho_e u_e^i \quad i-\text{th momentum density} \tag{17}$$

$$\int \frac{d^3p}{(2\pi)^3} p^{-1} p^i p^j f_\gamma(p_i) = \frac{1}{6}\rho_\gamma \Pi_\gamma^{ij} + \frac{1}{3}\rho_\gamma \delta^{ij} \tag{18}$$

Thus on simplification, the Euler equations for the two fluids reduce to

$$m_p n u_p^\mu u_{p;\mu}^i - e n u_p^\mu F_\mu{}^i = 0 \tag{19}$$

$$m_e n u_e^\mu u_{e;\mu}^i + e n u_e^\mu F_\mu{}^i = -\frac{4\sigma_T \rho_\gamma n}{3}\left[(u_e^i - u_\gamma^i) + \frac{1}{8} u_{ej} \Pi_\gamma^{ij}\right] \tag{20}$$

Here, for matter fields, we neglect pressure in comparison to energy density ($w = p/\rho$ for matter is 0). Also, on the basis of charge neutrality $n_e \sim n_p = n$. Defining u^μ, the center of mass four-velocity, and j^μ, the net current as

$$u^\mu = \frac{m_p u_p^\mu + m_e u_e^\mu}{m_e + m_p} \tag{21}$$

$$j^\mu = en(u_p^\mu - u_e^\mu) \tag{22}$$

one can combine equations (19) and (20) to give

$$\begin{aligned} -\frac{m_p m_e}{e}\left(n u^\mu \left(\frac{j^i}{n}\right)_{;\mu} + j^\mu \left(\frac{m_p - m_e}{m_p + m_e}\frac{j^i}{en} - u^i\right)_{;\mu}\right) & \\ + en(m_p + m_e) u^\mu F^i_\mu - (m_p - m_e) j^\mu F^i_\mu &= -\frac{4 m_p \sigma_T \rho_\gamma n}{3} \\ & \times \left((u_e^i - u_\gamma^i) + \frac{1}{8} u_{ej} \Pi_\gamma^{ij} \right) \end{aligned} \tag{23}$$

Now one can compare the relative size of each terms, to obtain the relevant terms. j^μ is proportional to $(u_p^\mu - u_e^\mu)$, by definition from equation (22). The difference in the proton and electron velocities in the plasma is very small. Therefore, since $|u^i| \gg |u_p^i - u_e^i|$, the third term on the left hand side of equation (23) ($(m_p - m_e) j^\mu F^i_\mu$) can be dropped, relative to the second term $en(m_p + m_e)u^\mu F^i_\mu$. Also, out of the 3 terms in the bracketed first term of the left hand side, the term quadratic in j^μ can be neglected, when compared to the other two terms scaling as $u^i j_\mu$. Since $F^{\mu\nu}_{;\nu} = j^\mu$, one can estimate that the ratio of the other terms in the first bracketed term of the LHS, to the second term of the LHS, scales as $\frac{c^2}{L^2 w_p^2} \sim 10^{-47}\left(\frac{10^{10}\mathrm{cm}^{-3}}{n}\right)\left(\frac{1\mathrm{Mpc}}{L}\right)^2$ where L is the characteristic length scale of the order ~ 100 pc, c is the speed of light, and $w_p = \sqrt{\frac{4\pi n e^2}{m_e}}$ is the plasma frequency.

This leaves the relation (23) as

$$u^\mu F^i_\mu = -\frac{4\sigma_T \rho_\gamma}{3e}\left((u_e^i - u_\gamma^i) + \frac{1}{8} u_{ej} \Pi_\gamma^{ij} \right) \tag{24}$$

Thus the photon push exerted on the electron field, and the anisotropic stress contribute to the presence of a magnetic field. Defining the combination $u^\mu F^i_\mu$ to be C^i, one can now employ the Bianchi identities for the electromagnetic field: $F_{[\mu\nu,\lambda]} = 0$, and the relation for comoving magnetic field $B^i = \epsilon^{ijk} F_{jk}/2$ to obtain

$$\begin{aligned} 0 &= \epsilon^{ijk} u^\mu F_{[jk,\mu]} \\ &= u^\mu B^i_\mu - \frac{1}{u^0}\epsilon^{ijk} C_j u^0_{\ k} + \epsilon^{ijk} C_{j,k} - (u^i_{\ ,j} B^j - u^j_{\ ,j} B^i) + \frac{u^0_{\ ,j}}{u^0}(B^j u^i - B^i u^j) \end{aligned} \tag{25}$$

One can now expand the fluid velocities, photon anisotropic stress and energy density perturbatively as

$$\rho_\gamma = \rho^{(0)}_\gamma + \rho^{(1)}_\gamma + \ldots, \qquad \Pi^{ij}_\gamma = \Pi^{ij\,(1)}_\gamma + \ldots$$
$$\mathrm{u}_0 = 1 + u^{(2)}_0 + \ldots, \qquad u_i = u^{(1)}_i + u^{(2)}_i + \ldots$$

B has a small magnitude, so we can ignore most of the higher order terms in the perturbative series, when we plug in the series for each variable into equation (25) to get,

$$\begin{aligned} \dot{B}^i &\sim -\epsilon^{ijk} C_{j,k} \\ &\sim \frac{4\sigma_T \rho^{(0)}_\gamma}{3e}\epsilon^{ijk}\Big[\frac{\rho^{(1)}_{\gamma,k}}{\rho^{(0)}_\gamma}(u^{(1)}_{ej} - u^{(1)}_{\gamma j}) + (u^{(2)}_{ej,k} - u^{(2)}_{\gamma j,k}) + \frac{1}{8}(u^{(1)}_{el,k}\Pi^{l\,(1)}_{\gamma j} + u^{(1)}_{el}\Pi^{l\,(1)}_{\gamma j,k})\Big] \end{aligned} \tag{26}$$

where $i, j, k, l = 1, 2, 3$, $\dot{B}$ implies differentiation with respect to cosmic time, and "$, i$" denotes taking the ordinary derivative with respect to x^i.

The equation can be solved in using Fourier expansion. Notice that the vorticity (curl of the velocity field) of the electron fluid contributes at the second order to the magnetic field, as the terms containing $u^{(2)}_{ej,k}$ demonstrate. Also, the equation above shows that the photon anisotropy contributes to the magnetic field.

Thus one can produce a magnetic field, based on the motion of the electron-proton-photon plasma, the existence of which is not dependent on the presence of any seed magnetic fields, and is only a function of the photon anisotropy, photon pressure on the electron flow, and second-order electron vorticity effects. Using scaling arguments, [1] estimates the order of magnitude of the field thus produced at a characteristic length scale of 10 Mpc to have been $\sim 10^{-19}$ G in the decoupling period, and assuming adiabatic decay with the expansion of the universe, to have a current value of $\sim 10^{-25}$G. Note that at smaller scales one would have to take the rapidly diffusive electron-proton Coulomb scattering term into account as well.

3 Bibliography

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