Cosmic Topology

Methods of Detection and Constraints from CMB Anisotropy

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1 Introduction

In this paper I will discuss the theory and practical applications of cosmic topology and its detection. In the standard FLRW cosmological model, we study space which we assume to be homogeneous and isotropic, and so it has constant curvature. However, the mathematical notion of curvature is a local property so there are many topologically distinct manifolds that satisfy this geometrical assumption. When these manifolds are positively or negatively curved, they are characterized by their geodesics. Therefore when we study the null geodesics of light, we have a chance of detecting the cosmic topology. This is certainly a worthwhile result in and of itself, but knowledge of the cosmic topology puts constraints on parameters of the mass-energy content theories. Hence detection of the cosmic topology could prove itself useful in the future for deciding the validity of competing theories such as ΛCDM, quintessence, and generalized Chaplygin gas (GCG).

This topic is a meeting point for many robust areas of mathematics and physics. Such areas range from differential topology and the classification of 3-manifolds to general relativity and the quest for understanding of "dark matter" and "dark energy." An entire text and more could be, and probably has been, written on this topic. So clearly this paper cannot address all of these subjects. What this paper *will* do is give a general background and then discuss the [1]. In the second section I will motivate and discuss the relevant mathematical concepts needed to make sense of this topic. In the third section I will give a brief overview of relevant background cosmology. In the fourth section I will discuss what the possible space topologies are and how they may or may not be detectable. In the fifth section I will discuss how cosmic topology can be used as a tool to restrict parameters of energy-matter theories. Finally, in the sixth section, I will discuss the paper I focused on most, that shows which cosmic topologies are consistent with the WMAP1 data.

2 Mathematical Necessities

From electrons orbiting a nucleus to the expanding universe, physics is largely concerned with the dynamical behavior of systems. The arena in which the system exists effects its behavior. Clearly a rolling ball constrained to the surface of a sphere will behave differently than one rolling along a plane. In order to properly understand the dynamics of a system, one must understand the space in which this system exists. Indeed Einstein showed that, in some cases, dynamical behavior can be viewed as the geometry of the space. So, for physicists, it is important to seek to understand the "shape" of a space. Two important mathematical issues have just been raised: what defines space and what defines shape.

What is a space? Well, as physicists, we view our spaces as arenas for systems, and so we have a few basics requests of our space:

1. We wish our space to have a notion of "nearness" by which we may study the positions of the objects in our system with respect to one another.
2. We wish our space to have a notion of "rate of change" by which we may study the motion within the system.

These two physical demands on our space can be articulated in well-defined mathematical objects. The first demand motivates the most general notion of a space: a set with some notion of "nearness," i.e. a topological space. To make this precise, let X be a set and define τ to be the collection of all open subsets of X. The pair (X, τ) defines a topological space and can be thought of as a set X on which τ endows a topology. A continuous map f between topological spaces (X, τ) and (Y, ψ) is a set map that preserves the structure of the topological space. More precisely, f is continuous if, given $V \in \psi$, then $f^{-1}(V) \in \tau$. If there exists a continuous map $g : Y \to X$ such that, $g \circ f = Id_X$, and $f \circ g = Id_Y$, then f is a topological space isomorphism (i.e. a homeomorphism). If X and Y are homeomorphic, then, as far as *topological spaces* are concerned, X and Y are the same. [2]

When dealing with topological spaces without additional structure, all we care about are classes of homeomorphic spaces. Any two members of a given class look the same in the eyes of a topologist. Now we have come across a major issue in topology: when are two spaces homeomorphic? This is easy to answer for simple spaces, like the donut and the coffee cup, but it gets hard, fast. An effective way to distinguish different spaces is to find a *topological invariant*, a property of a space that must be preserved by a homeomorphism. Although non-homeomorphic spaces may share such topological invariants, two spaces cannot be homeomorphic if they do not share topological invariants. The entirety of homotopy, homology, and cohomology theories are based on this concept. For more, see [3].

Although we now have a good notion of a space with "nearness," we still don't have the capability to measure this "nearness". In geometry, the way we measure distance between two points is by comparing their coordinates. So in order to measure distance in a space we need to have a well-defined grid on the space. Since the only grid in sight is $\mathbb{R}^n$, we need to associate $\mathbb{R}^n$ with regions of the space. Let (X, τ) be a topological space which is both Hausdorff and has a countable basis. We then define an n-manifold structure on X by insisting that around every point in X, there exists a neighborhood homeomorphic to $\mathbb{R}^n$. The collection of these neighborhoods and associated homeomorphisms comprise the *atlas* $\mathcal{A}$, which we call smooth if all the homeomorphism are sufficiently compatible. If $\mathcal{A}$ is a smooth atlas on X, then we call the triplet $(X, \tau, \mathcal{A})$ a differentiable manifold. Where (X, τ) is a topological space and $\mathcal{A}$ endows (X, τ) with local charts (i.e. a notion of differentiability). An important point to keep in mind here is that differentiable structure is defined locally.

Differentiable manifolds, like any good mathematical object, have a notion of maps between themselves. A map f of differentiable manifolds $(X, \tau, \mathcal{A})$ and $(Y, \psi, \mathcal{B})$, (i.e. a *smooth map*), is a continuous map that preserves the differentiable structure of X. More precisely, f is smooth at $x \in X$ if there exists a chart $(V_\alpha, \varphi_\alpha)$ in X about x and a chart (U_β, ϕ_β) in Y about $f(x)$ such that $\phi_\beta \circ f \circ \varphi_\alpha^{-1} : \varphi_\alpha \circ f^{-1}(U_\beta) \to \mathbb{R}^n$ is C^∞ as a map from a subset of $\mathbb{R}^n$ to a subset of $\mathbb{R}^n$. If there exists a smooth map $g : Y \to X$ such that $g \circ f = Id_X$, and $f \circ g = Id_Y$, then f is a differentiable manifold isomorphism (i.e. a *diffeomorphism*). If X and Y are diffeomorphic, then, as far as *differentiable manifolds* are concerned, X and Y are the same.

Although we now have a good notion of a space with a measure of "nearness," we cannot yet make sense of the "rate of change" of position. Suppose we have a manifold M with a Riemannian (or pseudo-Riemannian) metric g. An isometry of M is a diffeomorphism which preserves distance (i.e. it preserves the metric) and, hence, curvature. A diffeomorphism $f : M \to N$ is an isometry at $x \in M$ if, given $v, w \in T_xM$, then $g(v, w) = g(f^*v, f^*w)$. The collection of all isometries $f : M \to M$, denoted $Iso(M)$, is a group under composition. A subgroup $\Gamma < Iso(M)$ is discrete if it has finitely many elements. If $\varphi \in Iso(M)$ moves every point, i.e. $\varphi(x) \neq x$ for any $x \in M$, then we say that φ is fixed point-free.

So as physicists, there are really two relevant notions of "shape:" topological and geometrical. The topology of a space is a global statement, while geometry is inherently local.

3 Cosmology

Cosmology begins with General Relativity. General relativity attempts to describe the dynamical behavior of gravity. In so doing it relates curvature of the 4-dimensional space-time manifold $\mathcal{M}_4$ to its matter content. This connection is given by Einstein's equation:

$$R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi G T_{ab} \tag{1}$$

Certainly given g_{ab} or T_{ab} in a region of our 4-dimensional space-time manifold $\mathcal{M}_4$, we can use (1) to find the other. However, the real power of (1) is that it gives us the dynamical relationship between g_{ab} and T_{ab} when we already know both. To apply general relativity to the large scale structure of the universe, and use (1), we must first find g_{ab} and T_{ab} on all of $\mathcal{M}_4$. This could be a formidable task. Luckily observations of CMBR and other radiating sources suggest that the universe has a great deal of symmetry. Based on these observations, we assume that space is isotropic and homogeneous. Hence we view our 4-dimensional space-time manifold $\mathcal{M}_4$ as $\mathbb{R} \times \mathcal{M}_3$ where $\mathcal{M}_3$ is an isotropic and homogeneous 3-dimensional space manifold. This isotropy and homogeneity condition immediately implies that $\mathcal{M}_3$ has constant curvature. We conclude that $\mathcal{M}_3$ has the geometry of S^3, $\mathbb{R}^3$, or H^3. Consequently, the space-time metric for an expanding universe takes the form:

$$\mathbf{g_{ab}} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & a^2(t)S_k^2(\chi) & 0 \\ 0 & 0 & 0 & a^2(t)S_k^2(\chi)sin^2(\theta) \end{pmatrix} \tag{2}$$

Or equivalently, and more compactly:

$$ds^2 = -dt^2 + a^2(t)(d\chi^2 + S_k^2 d\Omega^2) \tag{3}$$

where:

$$S_k = \begin{cases} sin\chi & \text{if } k = 1 \text{ i.e. if curvature is positive} \\ \chi & \text{if } k = 0 \text{ i.e. if curvature is zero} \\ sinh\chi & \text{if } k = -1 \text{ i.e. if curvature is negative} \end{cases} \tag{4}$$

Additionally, the energy-momentum tensor T_{ab} must reflect the spatial homogeneity and isotropy. It follows that T_{ab} cannot have any non-diagonal components, and it must have the same components in each spatial direction, i.e.

$$\mathbf{T_{ab}} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & pa^2 & 0 & 0 \\ 0 & 0 & pa^2 & 0 \\ 0 & 0 & 0 & pa^2 \end{pmatrix} \tag{5}$$

Solving (1) yields the Friedman equations:

$$3\left(\frac{\dot{a}}{a}\right)^2 = 8\pi G\rho - \frac{3k}{a^2} \tag{6}$$

$$3\left(\frac{\ddot{a}}{a}\right) = -4\pi G(\rho + 3p) \tag{7}$$

where ρ is the energy density of the universe (fluid). Since we want to use these equations, we must determine ρ and p. We can calculate the total ρ and p by summing up the contributions made by the constituents of the universe. Classically we assume these to be (1) visible matter, (2) radiation, and (3) curvature.

Recent observations have brought many of these classical statements into question. In the following, we focus on what the constant curvature assumption really implies about $\mathcal{M}_3$ and what the density parameter Ω really says about the of constituents of the universe.

4 Detection of the Cosmic Topology

In the standard FLRW cosmological model, we insist that the spatial manifold $\mathcal{M}_3$ be homogeneous and isotropic, and as a necessary consequence, $\mathcal{M}_3$ has constant curvature. Then we usually say that $\mathcal{M}_3$ is either S^3, $\mathbb{R}^3$, or H^3 depending on whether $\mathcal{M}_3$ has positive, zero, or negative curvature respectively. This glosses over a subtle point: differentiable structure, and hence curvature, is defined *locally.* To say that $\mathcal{M}_3$ has positive curvature is not to say $\mathcal{M}_3$ *is* S^3, but rather that locally $\mathcal{M}_3$ is isometric to S^3.

The classification of all manifolds of constant curvature is still an open problem today. However, it is easy to find such a manifold which is not homeomorphic to S^3, $\mathbb{R}^3$, or H^3. Let $\widetilde{\mathcal{M}_3} = (S^3, \mathbb{R}^3, \text{or} H^3)$ and let Γ be a discrete, fixed-point free group of isometries of $\widetilde{\mathcal{M}_3}$. Then $\widetilde{\mathcal{M}_3}/\Gamma$ is a manifold of the same constant curvature as $\widetilde{\mathcal{M}_3}$. So our spatial manifold

Table 1: Discrete Isometries of S^3 and Their Properties

Group Name	Symbol	Order	Injectivity Radius
Cyclic	Z_n	n	π/n
Binary dihedral	D_m^*	4m	$\pi/2m$
Binary tetrahedral	T^*	24	$\pi/6$
Binary octahedral	O^*	48	$\pi/8$
Binary icosahedral	I^*	120	$\pi/10$

could really be one of an infinite collection of these quotient manifolds. It turns out that all homogeneous orientable manifolds with positive curvature are given by quotienting out by one of the groups in table 1. Unfortunately, isometry groups for hyperbolic manifolds are much more difficult to characterize, and thus hyperbolic manifolds have yet to be completely classified [A few understood hyperbolic manifolds are listed in table two].

Certainly the realization that $\mathcal{M}_3$ can be multiply connected is more than just an interesting subtly of the standard FLRW model. Suppose the space we live in is a 3-dimensional space-time with $\mathbb{R}^2$ as its 2-dimensional spatial slice. Since $\mathbb{R}^2$ has no curvature, geodesics are simply straight lines. Now consider the isometries $\mathcal{R} : \mathbb{R}^2 \to \mathbb{R}^2$ where $\mathcal{R}(x, y) \equiv (x + 1, y)$ and $\mathcal{U} : \mathbb{R}^2 \to \mathbb{R}^2$ where $\mathcal{U}(x, y) \equiv (x, y + 1)$. Let G be the subgroup of $Iso(\mathbb{R}^2)$ generated by $\mathcal{R}$ and $\mathcal{U}$. By construction, G is a discrete group, and fixed point free. Now $\mathbb{R}^2/G$ is precisely quotienting out by the $\mathbb{Z}^2$ lattice, and so $\mathbb{R}^2/G$ is none other than the torus $\mathbb{T}$. Hence the straight line $y = \frac{1}{2}$ gets projected to a *closed* line about the hole of the quotient manifold $\mathbb{T}$.

Continuing this thought experiment, suppose we live in $\mathbb{R}^2$ at the coordinate $(0, 0)$ and there is a star at the coordinate $(\frac{1}{4}, 0)$ which goes super nova. To reach us at $(0, 0)$, the light from this event will follow the straight line *left* along the x-axis. However, if we live in $\mathbb{R}^2/G$, the light from this event will *also* reach us at $(1, 0) = (0, 0)$ by following the geodesic defined by going along the x-axis to the *right*. The fact that there are two distinct geodesics from the star to us is already significant. In addition, there are two other subttlties n what is happening in this though experiment:

1. The same event will be observed by us multiple times in completely different regions of the sky
2. The multiple signals of the same event will have had to travel different distances to reach us, and hence we will observe the same event at different times (according to our watches).

Table 2: First Seven Manifolds in the Hodgson-Weeks Census of Closed Hyperbolic Manifolds

Manifold	Volume	Injectivity Radius
m004(1,2)	1.398	0.183
m004(6,1)	1.284	0.240
m003(-4,3)	1.264	0.287
m003(-2,3)	0.981	0.289
m003(-3,1)	0.943	0.292
m009(4,1)	1.414	0.397
m007(3,1)	1.015	0.416

Based on this notion, we expect that if $\mathcal{M}_3$ were in fact a quotient manifold, then given two points there would exist more than one geodesics joining them. (Certainly this is also true in the non-quotient case when we have mechanisms such as gravitational lensing, but these effects are different as in the former the effect is global, while in the latter, the effect is localized to small regions.) Equivalently, $\mathcal{M}_3$ would possess closed geodesics. For the case when we have nontrivial curvature, Mostow proved that these closed geodesics actually describe the topological structure of M_3 [9].

Theorem *(Mostow)* The lengths of the closed geodesics in a compact orientable manifolds with nonzero constant curvature are a topological invariant.

This result is just what we were looking for. It provides us with a topological invariant that we can use to distinguish between possible $\mathcal{M}_3$'s. All we have to do is observe some sign of closed geodesics and we we will be able to calculate the cosmic topology. Let l_s be the length of the smallest closed geodesic, and define the *injectivity radius* r_{inj} as $\frac{l_s}{2}$. To observe *any* sign of nontrivial cosmic topology, we have to be able to observe the light emitted from one event that has travelled to us along two *different* geodesics, and hence we must be able to see *at least* as far as half the length of the smallest closed geodesic. [Note that 2 geodesics from a source compose to create a closed geodesic in space.]

In the FLRW universe with metric (4) and nonzero curvature, $a(t)$ can be identified as the curvature radius of the universe. Then χ is the distance (in the units of $a(t)$) of the point (χ, θ, ϕ) from the origin of $\widetilde{\mathcal{M}_3}$, which we take to be our position. As a consequence of the causal structure of the standard model, we can only survey our universe to some finite depth, say $\chi_{obs}(u_i, z_{obs})$ which is a function of *both* the parameters of our theory (u_i) and the red-shift z_{obs}. More precisely, to be able to observe any sign of nontrivial topology, it is necessary that $\chi_{obs} > r_{inj}$.

If it turns out that $\chi_{obs} > r_{inj}$ then we are unable to observe nontrivial cosmic topology, not as a result of insufficient technology, but rather as a result of the causal structure of space-time. Furthermore, even if $\chi_{obs} > r_{inj}$ we still have to deal with the fact that

1. Two images of a given cosmic object have (likely) travelled different distances and hence corresponds to different periods in its life, making their identification difficult.
2. Two images of a given cosmic object are (likely) seen at different angles, make morphological identification difficult.
3. High obscurations could easily hide images.

Even though some people [10] seem optimistic about seeing patterns in the CMB indicative of being multiply connected, taking all the above factors into account, I think it is rather unlikely that we will ever directly detect the cosmic topology. This is an unfortunate, yet unavoidable, reality when it comes to experimental sciences of this nature. However, as shall be discussed further in section 6, we may still indirectly find evidence of cosmic topology.

5 Using the Cosmic Topology: Energy-Matter Content of the Universe

Current observations and theory tell us we are in a matter dominated regime where both the radiation and curvature contributions to Ω are negligible. Surprisingly, when we observed that $\Omega_{m0} \sim 0.05$ we also found that a total energy parameter $\Omega_0 \sim 1$. These observations tell us one of two things:

1. The Friedmann equations do not accurately represent the large scale universe

   or

2. The Friedmann equations *do* accurately represent the large scale universe, but we made a mistake in our assumptions on what composes the universe.

Since there are plenty of reasons to believe that the Friedmann equations do accurately represent the large scale universe, we have to draw the latter conclusion. While searching for the missing Ω contributors, we "discover" the gravitational effects of non-visible matter (dark matter, or DM) with $\Omega_{DM} \sim 0.25$. Although this is progress, we still need to account for a whopping 0.7. The source of this 0.7, which we call dark energy (DE), must evenly permeate space since we have not yet been able to detect it. No physical model made prior to these revelations gives satisfactory explanations for these unaccounted for contributors, so new models are being to developed. A few of the leading models are ΛCDM, quintessence, and Generalized Chaplygin gas (GCG). Currently there is no strong reason to support one theory over another. Verification of any of these models ultimately rests upon future observational data. Currently WMAP and SNIa data place strong constraints on model parameters, and it is hoped that additional constraints from another data source would be enough to determine one model to be more accurate than the others. [7] and [8] show that the detection of the cosmic topology (trivial or otherwise) would further constrain the parameters of these model. Below I outline the argument from [7] showing how cosmic topology constrains the parameters of the CGC model. Similar arguments for ΛCDM are carried out in [8].

For simplicities' sake, it would be nice if the *two* unknowns, DM and DE, were really just *one* unknown. Perhaps there is a scalar field that at early periods of the universe, would behave as dust-like matter, while at later times would behave as a cosmological constant. CGC is such a field. Classically, the Chaplygin gas model describes a fluid with the equation of state:

$$p_{ch} = -\frac{A}{\rho_{ch}} \tag{8}$$

where A is some positive constant that is characteristic of the specific fluid being studied. Plugging this equation of state into Einstein's equation in an expanding universe, we see that the density evolution of this fluid as the universe expands is given by:

$$\rho_{ch} = \sqrt{A + \frac{B}{a^6}} \tag{9}$$

where B is an integration constant. Thus

$$\rho_{ch} \propto \begin{cases} a^{-3} & \text{if } t \ll 1 \\ a^{0} & \text{if } t \gg 1 \end{cases} \tag{10}$$

This means *precisely* that at early periods CG, behaves as dust-like matter, while at later times it behaves as a cosmological constant. We can generalize this further to the *Generalized Chaplygin Gas (GCG)* model with an aptly placed α. More precisely, consider the equation of state:

$$p_{ch} = -\frac{A}{\rho_{ch}^{\alpha}} \tag{11}$$

where α is some positive constant which is typically, but not necessarily, within $(0,1]$. When this equation of state is plugged into Einstein's equation in an expanding universe, we see that the density evolution of this fluid as the universe expands is given by:

$$\rho_{ch} = \left[A + \frac{B}{a^{3(1+\alpha)}}\right]^{\frac{1}{1+\alpha}} \tag{12}$$

In the GCG model, we assumed that the entirety of the [non-negligible] energy density is supplied by baryonic matter and GCG (i.e. $\Omega = \Omega_b + \Omega_{cp}$). Under this assumption, the Friedmann equation takes the form:

$$H^2 = \frac{8\pi G}{3}(\rho_b + \rho_{ch}) - \frac{k}{a^2} \tag{13}$$

If we further assume that the mass components do not interact, we find that the two energy densities evolve as a function of $a(t)$ in the following way:

$$\rho_b = \rho_{b0}\left(\frac{a_0}{a}\right)^3 \tag{14}$$

$$\rho_{ch} = \rho_{ch0}\left[(1-A)\left(\frac{a_0}{a}\right)^{3(1+\alpha)} + A\right]^{\frac{1}{1+\alpha}} \tag{15}$$

Noting that redshift is given by $z = \frac{a_0}{a} - 1$, we use the above energy density terms to rewrite the Friedmann equation in terms of the degrees of freedom (A, α, and z) within our theory.

$$H(z) = H_0\left[\Omega_{ch0}\left[A + (1-A)(1+z)^{3(1+\alpha)}\right]^{\frac{1}{1+\alpha}} + \Omega_{b0}(1+z)^3 + (1-\Omega_0)(1+z)^2\right]^{\frac{1}{2}} \tag{16}$$

where $\Omega = \Omega_{b0} + \Omega_{cp0}$. Using the above equation we now derive the redshift-distance relationship for the GCG theory for the case in which curvature is non-zero.

$$H(z) = H_0\left[\Omega_{ch0}\left[A + (1-A)(1+z)^{3(1+\alpha)}\right]^{\frac{1}{1+\alpha}} + \Omega_{b0}(1+z)^3 + (1-\Omega_0)(1+z)^2\right]^{\frac{1}{2}} \tag{17}$$

where $x = z + 1$ and, as we are assuming $k \neq 0$, we identify the curvature radius with the scalar factor, which is given by

$$\chi(A, \alpha, z) = \sqrt{|1 - \Omega_0|} \int_1^{1+z} \left[\Omega_{ch0} \left[A + (1 - A) x^{3(1+\alpha)} \right]^{\frac{1}{1+\alpha}} + \Omega_{b0} x^3 + (1 - \Omega_0) x^2 \right]^{-\frac{1}{2}} dx \tag{18}$$

This gives us exactly what we wanted: the observation depth as a function of the parameters of the GCG theory. It is with this equation that we may answer the question whether GCG is consistent with nontrivial cosmic topology. Let M be a possible space manifold candidate and r_{inj}^M be its injectivity radius. Consider the following function:

$$\chi(A, \alpha, z) = r_{inj}^M \tag{19}$$

Then this equation defines a region of parameters for which cosmic topology is detectable. If we detect cosmic topology, then we may rule out all parameter models which predict that we should not be able to observe cosmic topology. Hence, given an observation of cosmic topology, (18) constrains the parameters A and . When used in conjunction with data on Ω_0 from COBE and WMAP, this heavily constrains these parameters.

Unfortunately, as described in the previous section, there is currently no evidence for a certain cosmic topology, and hence the papers [7] and [8] are unable to make any specific predictions. Regardless, these papers show how cosmic topology is intertwined with matter-energy theories and the density parameter Ω.

6 CMB Anisotropies And The Reason We Observe Suppressed Low Modes

Although the CMB is overwhelmingly isotropic, which is one of the strongest observational reasons to assume that $\mathcal{M}_3$ is isotropic and homogeneous, there are small temperature fluctuations, $\delta T(\theta, \phi)$, as one sweeps across the sky. However, both WMAP 1 and COBE show slightly less anisotropy than would be expected from the standard inflationary model. Usually to remedy such problems requires some fine tuning of the inflationary parameters. However, [1] shows that instead of fine tuning inflation, if we assume space $\mathcal{M}_3$ to be in a certain class of multiply connected spherical manifolds, then these modes are *naturally* suppressed.

The WMAP 1 data set gives a prediction that $\Omega = 1.02 \pm 0.02$ (with errors given b the $1^{st}\sigma$ deviation uncertainty). Assuming the validity of these results, spatial curvature is positive. For the rest of the paper, we shall then assume that $\mathcal{M}_3 = S^3/\Gamma$ where Γ is a fixed point free discrete subgroup of $SO(4)$ (and may be trivial) [see table 1]. Let $q : S^3 \to \mathcal{M}_3$ be the quotient map.

Of the several theoretical contributors to this anisotropy, the most significant is that of the Sachs-Wolfe effect. This effect is the result of gravitational perturbations present at the last scattering surface (LSS) which have been propagating since. Thus, to understand the evolution of this perturbation with the expansion of the universe, we have to understand how waves propagate through space. Define $\psi_\beta^{\mathcal{M}_3}$ to be the eigenfunctions on $\mathcal{M}_3$ and $E_\beta^{\mathcal{M}_3}$ to be the associated eigenvalues indexed by β were β is the wave number. The vibrations at a point $\hat{p} \in S^3$ are given by regular solutions to the generalized Helmholtz equation:

$$(\triangle + E_\beta^{\mathcal{M}_3})\psi_\beta^{\mathcal{M}_3}(\hat{p}) = 0 \tag{20}$$

where the Laplace-Beltrami operator $\triangle$ on S^3 is given by

$$\triangle f = \frac{1}{\sqrt{|g|}}\nabla_i \sqrt{|g|}\nabla^i f \tag{21}$$

and g is the Riemannian metric on S^3.

As the geometry of S^3 and $\mathcal{M}_3$ are the same, (20) holds on both. However, as $q(\hat{p}) = q(g \cdot \hat{p}) := p$ for all $\hat{p} \in S^3$ and $g \in \Gamma$, the Helmholtz equation on $\mathcal{M}_3$ comes with the additional periodicity requirement:

$$\psi_\beta^{\mathcal{M},i}(g \cdot p) = \psi_\beta^{\mathcal{M},i}(p) \tag{22}$$

The spectrum on $\mathcal{M}_3$ is discrete, and the eigenvalues can be expressed in terms of the wave number $\beta \in \mathbb{N}$ as $E_\beta = \beta^2 - 1$, and are independent of the degeneracy index $i = 1, 2, \ldots, r_\beta^{M_3}$, where $r_\beta^{M_3}$ denotes the multiplicity of the mode β. The important point is that, depending on $\mathcal{M}_3$, $\beta does not take all values in \mathbb{N}. See [1] for a chart of the attained values.$

Now given any wave equation ψ on $\mathcal{M}_3$, we may may lift it to the wave equation $\hat{\psi}$ on $\S^3$ where we define $\hat{\psi}(\hat{p}) = \psi(q(\hat{p}))$ for all $\hat{p} \in S^3$. As Γ is a group of fixed point free isometries, it follows that $\hat{\psi}$ is well defined. The lift of the eigenfunction $\psi_\beta^{\mathcal{M},i} on \mathrm{M}_3$ can be expanded in terms of the eigenfunctions $\psi_\beta^{S^3}$ on S^3

$$\hat{\psi}_\beta^{\mathcal{M},i}(\hat{p}) = \sum_{l=0}^{\beta-1}\sum_{m=-l}^{l} \xi_{\beta lm}^i(\mathcal{M})\psi_\beta^{S^3}(\hat{p}) \tag{23}$$

As we view our observable sky as the surface of S^2, we expand our temperature observations as spherical harmonics. By excluding the monopole ($l = 1$) and dipole terms ($l = 2$), we then find an expression for the temperature fluctuations.

$$\delta T(\theta, \phi) := \sum_{l=2}^{\infty} \sum_{m=-l}^{l} a_{lm} \tilde{Y}_{lm}(\theta, \phi) \tag{24}$$

When we average the real expansion coefficients a_{lm} over primordial perturbations, we get the multipole moments

$$C_l := \frac{1}{2l+1} \left\langle \sum_{m=-l}^{l} (a_{lm})^2 \right\rangle \tag{25}$$

and from this, the angular power spectrum

$$\delta T_l^2 := \frac{l(l+1)}{2\pi} C_l \tag{26}$$

Putting these terms together yields the Sachs-Wolfe contribution to the multipole moment for $\mathcal{M}_3$:

$$C_l^{SW}(\mathcal{M}_3) = \frac{N}{9} \sum_{\beta > l}{}' \frac{r^{\mathcal{M}_3}}{\beta^2} P_\Phi(\beta) g_\beta^2(\eta_{LSS}) R_{\beta l}^2(\tau_{LSS}) \tag{27}$$

Where the R denotes a radial term. The significance is that, as β does not hit all natural numbers for a given manifold $\mathcal{M}$. Hence, the Sachs-Wolfe contribution is naturally lower for multiply connected manifolds. Upon more careful analysis, comparing the observed anisotropies with each of the possible multiply connected manifolds, [1] concludes that only two are consistent, S^3/O^* and S^3/I^*

7 Conclusion

Comic topology is a fascinating topic that mixes mathematical theories with physical law [10]. Although it is possible, it is unlikely that we will be able to directly observe cosmic topology. However, it is possible to observe cosmic topology indirectly, such as through the anisotropies of the CMB [1] [12]. Upon understanding cosmic topology, we will better understand energy-matter theories [7] [8], and then perhaps, a better understanding of what the mechanism is that drives the universal acceleration today.

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