Direct Detection of the Inflationary Gravitational Wave Background

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June 2, 2006

The paper I am writing about is titled "Direct detection of the inflationary gravitational wave background". The authors are Tristan L. Smith, Mark Kamionkowski, and Asantha Cooray. The paper was submitted for publication on June 17, 2005, and finally published January 9, 2006, in Physical Review D.

The paper discusses the prospects of directly detecting gravitational waves that resulted from inflation using the proposed BBO (Big Bang Observer) and DECIGO (Deci-Hertz Interferometer Gravitational-wave Observer) satellite missions. The authors find that direct detection may be possible, and, furthermore, that direct detection may allow us to extract data about inflation not accessible to CMB (Cosmic Microwave Background) and LSS (Large Scale Structure) experiments.

1 Physics Preliminaries

Before discussing the meaty points of the paper, it is useful to review the basic equations that the analysis depends on. For the time being, we will be assuming an isotropic, homogeneous, and flat universe. The metric of such a universe is given by

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2) \ . \tag{1}$$

The function $a(t)$ is known as the scale factor. We define Hubble's constant, H in terms of the scale factor as

$$H \equiv \frac{\dot{a}}{a} \ . \tag{2}$$

Recall the Friedman equation,

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \ , \tag{3}$$

where ρ is the energy density, and we have assumed zero curvature. (The curvature term is not important, because it will quickly tend to zero during inflation.) Differentiating this equation with respect to time, t, we obtain

$$\frac{d(H^2)}{dt} = 2H\dot{H} = 2H\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) = 2H\left(\frac{\ddot{a}}{a} - H^2\right) = \frac{8\pi G}{3}\dot{\rho} \ . \tag{4}$$

We may remove the dependence on $\dot{\rho}$ by utilizing conservation of energy: $d(\rho V) + pdV = 0$, where V is a physical volume. This implies that

$$\begin{aligned} &\Rightarrow d(\rho a^3) + pd(a^3) = 0 \\ &\Rightarrow d\rho \cdot a^3 + 3a^2\rho \cdot da + 3a^2 p \cdot da = 0 \\ &\Rightarrow d\rho + 3\rho\frac{da}{a} + 3p\frac{da}{a} = 0 \\ &\Rightarrow \dot{\rho} + 3H(\rho + p) = 0 \ , \end{aligned}$$

giving us the continuity equation for our metric. Inserting this into (4) and using (3) gives us

$$2H\left(\frac{\ddot{a}}{a} - \frac{8\pi G}{3}\rho\right) = -8\pi G H(\rho + p) \ .$$

Rearanging terms, we obtain

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \ . \tag{5}$$

In order for rapid expansion to occur, we require that $\ddot{a} > 0$. Therefore, the condition for inflation is that $\rho + 3p < 0$, or $\rho < -3p$. The lagrangian for a scalar field ϕ is given by

$$L = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) \ . \tag{6}$$

From this we can obtain the stress-energy tensor, and find that for a spatially homogeneous field,

$$\begin{aligned} \rho &= \frac{1}{2}\dot{\phi}^2 + V(\phi) \\ p &= \frac{1}{2}\dot{\phi}^2 - V(\phi) \ . \end{aligned}$$

Plugging these values into (5) we obtain

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3}(\dot{\phi}^2 - V(\phi)) \ . \tag{7}$$

Therefore, for inflation to be driven by the field ϕ, we require that $\dot{\phi}^2 < V(\phi)$. Using the lagrangian density (6) and noting that the action is given by

$$S = \int L\sqrt{-g}\ d^4x \tag{8}$$

we can find the equation of motion of the field:

$$\frac{\partial L}{\partial \phi} = \frac{\partial}{\partial x^\mu} \frac{\partial L}{\partial(\partial_\mu \phi)}$$

which becomes

$$\ddot{\phi} + 3H\dot{\phi} - \nabla^2 \phi + dV/d\phi = 0 \ . \tag{9}$$

If $\dot{\phi}^2 << V(\phi)$, then we will have inflation. This leads to slow-roll aproximation, in which we can neglect the $\ddot{\phi}$ term in the equation of motion. Finally, for a homogeneous field we have

$$3H\dot{\phi} + dV/d\phi = 0 \ . \tag{10}$$

In order for this equation to be valid, a necessary condition is that the slow roll parameters,

$$\epsilon \equiv \frac{M_{pl}^2}{16\pi}\left(\frac{V'}{V}\right)^2 \ , \tag{11}$$

$$\eta \equiv \frac{M_{pl}^2}{8\pi}\frac{V''}{V} \ , \tag{12}$$

$$\xi \equiv \frac{M_{pl}^4}{64\pi^2}\frac{V'V'''}{V^2} \ . \tag{13}$$

are small compared with unity.

The main results of the paper deal with the gravity waves produced by inflation. The way this happens is by quantum fluctuations in the inflationary potential which then induce fluctuations in the energy density (scalar perturbations to the space-time metric). These in turn produce gravitational waves (tensor perturbations to the metric). These perturbations can be detected as polarization in the CMB. Gravitational waves induce a polarization that has a curl, and the scalar perturbations result in a curl-free polarization field.

We can compute the variance in the field by considering a perturbation in which $\phi \to \phi + \delta\phi$. If we consider plane wave expansion of the perturbation, with $\delta\phi = Ae^{ik^\mu x_\mu}$, and consider V to be approximately constant, we get the following equation for the perturbation:

$$\ddot{\delta\phi} + 3H\dot{\delta\phi} + \frac{k^2}{a^2}\delta\phi = 0 \ . \tag{14}$$

This equation is similar to the simple harmonic oscilator of quantum mechanics, except that it has an additional term. This theory can be quantized in a similar manner, using creation and annihilation operators:

$$\phi_k = \omega_k a_k + \omega_k^* a_k^\dagger \ . \tag{15}$$

In order to determine the fluctuations about the vacuum state, we need to determine the ω_k's. The solution (which can be checked by plugging into (14)) is given by

$$\omega_k = a^{-3/2}(2k/a)^{-1/2} e^{-ik/aH}(1 + iaH/k) \ . \tag{16}$$

When $aH/k >> 1$ (the horizon is much smaller than the wavelength), we have $\omega_k \rightarrow a^{-3/2}(2k/a)^{-1/2} \cdot iaH/k = iH/\sqrt{2k^3}$. This implies that the fluctuations about the vacuum are given by

$$\langle 0||\phi_k|^2|0\rangle = |\omega_k|^2 = \frac{H^2}{2k^3} \quad . \tag{17}$$

From here, we see that the fluctuations per decade are constant:

$$\begin{aligned} d(\delta\phi)^2 &= 4\pi k^2 \langle 0||\phi_k|^2|0\rangle dk/(2\pi)^3 \\ &= \frac{4\pi k^3}{(2\pi)^3}\langle 0||\phi_k|^2|0\rangle d\ln k \\ &= \left(\frac{H}{2\pi}\right)^2 d\ln k \ . \end{aligned}$$

Therefore,

$$\frac{d(\delta\phi)^2}{d\ln k} = \left(\frac{H}{2\pi}\right)^2 \quad . \tag{18}$$

The above gives the flavor of determining the perterbative fluctuations. Similar calculations can be done to determine the tensor perturbations and scalar perturbations to the metric. In the end, it turns out that the power spectra are given by

$$P_s(k) \approx \frac{128\pi}{3M_{pl}^6}\frac{V^3}{V'^2}\bigg|_{k=aH} \quad , \tag{19}$$

$$P_t(k) \approx \frac{128}{3}\frac{V}{M_{pl}^4}\bigg|_{k=aH} \quad . \tag{20}$$

Note that the tensor-to-scalar power ratio is given by $r \equiv 16\epsilon$.

The power spectra can be expanded about a wavenumber k_0 as

$$P_s(k) \approx P_s(k_0)\left(\frac{k}{k_0}\right)^{1-n_s+(\alpha_s/2)\ln(k/k_0)} \quad , \tag{21}$$

$$P_t(k) \approx P_t(k_0)\left(\frac{k}{k_0}\right)^{n_t+(\alpha_t/2)\ln(k/k_0)} \quad , \tag{22}$$

where

$$n_s(k) \simeq 1 - 6\epsilon + 2\eta, \tag{23}$$

$$n_t(k) \simeq -2\epsilon, \tag{24}$$

$$\alpha_s(k) \simeq 16\epsilon\eta - 24\epsilon^2 - 2\xi, \tag{25}$$

$$\alpha_t(k) \simeq 4\epsilon\eta - 8\epsilon^2 \ . \tag{26}$$

From the power spectrum $P_t(k)$ we can determine the amplitudes of the gravitational waves, $\langle |h_k|^2\rangle$ when the wavelenghth is larger than the horizon. When the wavelength becomes smaller than the horizon, the amplitude is redshifted as we would expect for radiation. Since $\rho \sim k^2 h_k^2 \propto a^{-4}$, $h_k(t_0) = h_k(t_k)(a(t_k)/a(t_0))$ where t_k is the time when the mode enters the horizon and t_0 is the time today. Therefore, the transfer function is given by $T(k) = a_k/a_0$. This function can be found by equating $a_0 k = a_k H_k$, noting that $g_{*S} a^3 T^3$ is constant, and that $H \simeq 1.66 g_*^{1/2} T^2/M_{pl}$. Putting these together yields

$$T(k) = 2.1 \times 10^{-20} \left(\frac{k}{6.5 \times 10^1 3Mpc^-1} \right)^{-1} (g_*(T_k)/100)^{-1/6}) \ , \tag{27}$$

from which we find

$$\langle |h_k|^2 \rangle^{1/2} = P_t(k)^{1/2} T(k) \ . \tag{28}$$

Another important concept in inflationary cosmology is the number of e-foldings, or the natural logarithm of the ratio of the scale factor at the end of inflation to the scale factor at the begining of inflation. It is simple to determine the number of e-foldings in the slow-roll limit. Since $H = \dot{a}/a = \sqrt{8\pi G/3} V^{1/2}$, we have $\ln a = \int \sqrt{8\pi G/3} V^{1/2} dt$. Also, $3H\dot{\phi} = -V'$, which implies that $dt = -\frac{3H}{V'} d\phi$. Putting all these pieces together, we get

$$N = \int H \, dt = -8\pi G \int \frac{V}{V'} \, d\phi \ , \tag{29}$$

for the number of e-foldings.

Some inflationary potentials $V(\phi)$ have special names. Among these are

Power Law Inflation: $V(\phi) = V_0 e^{-p\phi/M_{pl}}$

Chaotic Inflation: $V(\phi) = V_0 \left(\frac{\phi}{M_{pl}} \right)^{\alpha}$

Symmetry Breaking Inflation: $V(\phi) = V_0 \left[1 - \left(\frac{\phi}{\nu} \right)^2 \right]^2$

Hybrid Inflation: $V(\phi) = V_0 \left[1 + \left(\frac{\phi}{\mu} \right)^2 \right]$.

2 BBO and DECIGO

The proposed BBO and DECIGO are space based laser interferometers designed to detect gravitational waves with frequencies in the decihertz to several hertz ranges. These detectors will detect the movement of component satellites relative to each other by using a Michelson interfermeter. According the NASA's official BBO web page (http://universe.nasa.gov/program/bbo.html):

> The Big Bang Observer has the goal of direct detection of quanta of the gravitational field created during inflation. This could give us a direct view of the creation of space and time and, in combination with results from the Inflation Probe, determine the nature of the vacuum at energies far higher than we can hope to reach with ground-based accelerators.

The Big Bang Observer will reach this goal by identifying (and subtracting) the gravitational wave signals from every merging neutron star and stellar-mass black hole in the Universe. Measurement of these merger signals will directly determine the rate of expansion of the Universe as a function of time, extending the results of the Dark Energy Probe.

The Big Bang Observer can also pinpoint gravitational waves from the formation or merger of intermediate mass black holes. These are believed to form from the first massive stars born in our Universe. They will also enable even finer measurements of the structure of spacetime around black holes than will be possible with LISA.

Right now, the plan is to have BBO working by 2025.

The DECIGO is a similar project planned by Japan, except that the range of frequencies that it will measure will be larger (milli-hertz to 100 hertz). The launch date for DECIGO is 2024.

3 Main Results

The paper analyzes various inflation models, including power law inflation, chaotic inflation, symmetry breaking inflation, and hybrid inflation. Before we can get into the specific analyses, we must note some constraints on the parameters. First of all, at the wavenumber $k_0 = 0.05$ Mpc^{-1} (CMB/LSS scales), we have $P_s(k_0) = (2.45 \pm 0.23) \times 10^{-9}$, $n_s = 1.0 \pm 0.1$, and $|\alpha_s| < 0.04$. In addition, the authors of the paper take an upper limit of $r < 1$ for the tensor-to-scalar ratio. The authors take a frequency of 0.1 Hz, corresponding to $k = 6.47 \times 10^{13}$ Mpc^{-1} as the nominal frequency of the proposed detectors (BBO or DECIGO). From this this we see that $\Delta N = \ln(6 \times 10^{13}/0.05) \simeq 35$ e-folds of inflation separate the CMB/LSS and BBO/DECIGO scales.

3.1 Power-Law Inflation

The potential for power-law inflation is given by $V(\phi) = V_0 e^{-p\phi/M_{pl}}$. From this, we see that $\epsilon = p^2/(16\pi)$, $\eta = p^2/(8\pi)$, which imply that $n_s = 1 - 2\epsilon$. Given our constraint above on n_s, these relations give $\epsilon < 0.05$. Now, $\Delta N = \frac{8\pi}{p}\frac{\phi_g - \phi_c}{M_{pl}} \simeq 35$, where ϕ_g is the field at BBO scales and ϕ_c is the field at CMB scales. Therefore, we find that $P_t(k_g)/P_t(k_c) = e^{-2\epsilon\Delta N}$. From here we can determine the gravitational wave amplitude as a function of ϵ and maximize it to obtain 5.68×10^{-16}. From figure 3 (page 6) of the paper, this is detectable with BBO if $r > 10^{-3}$ and DECIGO if $r > 10^{-6}$.

3.2 Chaotic Inflation

A similar analysis holds for chaotic inflation, which has a potential given by

$$V(\phi) = V_0 \left(\frac{\phi}{M_{pl}}\right)^{\alpha}. \tag{30}$$

From this potential, we find that

$$\epsilon = (\alpha^2/16\pi)(M_{pl}/\phi)^2. \tag{31}$$

Therefore, inflation ends when $\phi = \alpha M_{pl}/(4\sqrt{\pi})$. N_e e-foldings of inflation between CMB horizon exit and the end of inflation constrain

$$\phi_c^2 = (M_{pl}^2/16\pi)(4\alpha N_c + \alpha^2). \tag{32}$$

(Using the e-foldings equation.)

$$\eta = \alpha(\alpha - 1)(M_{pl}/\phi)^2/(8\pi). \tag{33}$$

Therefore, $n_s = 1 - 2(\alpha + 2)/(4N_e + \alpha)$. After specifying α and N_c, the potential prefactor can be determined, by using the formula for the scalar power spectrum given in the preliminary physics section. The result is

$$V_0 = \frac{3\alpha^2 P_s(k_c)}{128\pi}\left(\frac{16\pi}{4\alpha N_e + \alpha^2}\right)^{(\alpha+2)/2} M_{pl}^4. \tag{34}$$

From here we may determine the gravitational wave amplitude,

$$\frac{128}{3} A_{GW} \frac{V_0}{M_{pl}^4}\left(\frac{\phi_g}{M_{pl}}\right)^{\alpha}. \tag{35}$$

Now, applying the constraints on r and n_s, we find that the gravitational waves are detectable with BBO if $r > 10^{-3}$ and DECIGO if $r > 10^{-6}$.

3.3 Symmetry Breaking Inflation

Symmetry breaking inflation has a potential given by $V(\phi) = V_0\left[1 - \left(\frac{\phi}{\nu}\right)^2\right]^2$. A similar analysis to the above concludes that the waves will be seen by both detectors (BBO and DECIGO).

3.4 Hybrid Inflation

Hybrid inflation has a potential given by $V(\phi) = V_0\left[1 + \left(\frac{\phi}{\mu}\right)^2\right]$. A similar analyisis is inconclusive on the detectability of the gravitational waves from inflation by DECIGO and BBO.

3.5 Broken Scale-Invariant Spectrum

In this part of the paper, the authors demonstrate that the BBO or DECIGO can provide information about the inflationary potential that will be difficult for CMB/LSS experiments to observe. They consider a broken scale-invariant potential,

$$V(\phi) = V_0 \times \begin{cases} (1 + cA\phi), & \phi < 0 \\ (1 + A\phi), & \phi \geq 0 \end{cases} \tag{36}$$

In order to analyze this potential, the authors assume that ϕ_c, the field value at CMB/LSS scales, is $N_0 = 10$ e-folds from the kink at $\phi = 0$. Next, they use the normalization of the scalar power spectrum to determine V_0:

$$V_0 = \frac{3P_s(K_c)}{128} \frac{A^2 M_{pl}^2}{(1 + A\phi_c)^3} M_{pl}^4 \ . \tag{37}$$

Using the equation for the number of e-folds in terms of potential and starting and ending field values, the equation is then integrated and set to 35, in order to constrain ϕ_g. Inflation ends when $\phi = M_{pl}/(4\sqrt{\pi}) - 1/(cA)$. This results in a constraint

$$cA < \frac{4\sqrt{\pi}}{M_{pl}\sqrt{1 + 4(N_g - N_0)}} \ . \tag{38}$$

Now the gravitational wave amplitude is given by

$$4A_{GW} P_s(k_c) A^2 M_{pl}^2 \frac{\sqrt{A^2 c^2 M_{pl}^2 (N_0 - N_g) + 4\pi}}{(A^2 M_{pl}^2 N_0 + 4\pi)^{3/2}} \tag{39}$$

where $A_{GW} = 2.74 \times 10^{-6} g_{100}^{-1/3}$. ($g_{100} = g_*(T_k)/100$, T_k is the temperature when mode k entered the horizon.) Since the amplitude depends on c, the gravitational waves will give us data about the potential near ϕ_g, while the CMB/LSS experiments will give us data about the potential near ϕ_c.

4 Conclusion

The authors demonstrated that inflation may be detectable with the BBO and/or DECIGO satellites. They have also shown that in some cases the BBO/DECIGO satellites may provide information about the potential that is not accessible to CMB/LSS experiments. In addition, direct detection of the inflationary gravitational wave background would provide evidence that the curl component of the CMB polarization is due to inflation and not other effects. The direct detection would provide evidence of inflation on vastly different distance scales, and potentially evidence of the scale-invariance of the gravity waves. Even though there is plenty of room in the parameter space consistent with current CMB/LSS data for the BBO/DECIGO missions to be failures, the potential of great discoveries with these missions probably overwhelms the risks.

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