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125c, Lecture Seventeen: 5/31/17

Quantum Field Theory (QFT)

(but today we'll only cover special relativity.)

We've been thinking about systems with $\dim \mathcal{H}$ = small. Let's consider the opposite extreme: $\dim \mathcal{H} = \infty$. True even for a single non-relativistic particle, but our current interest is QFT.

History / Motivation

Special relativity came along in 1905, so by the time QM was developed in the 1920's everyone knew QM would have to be made relativistic. Problem: Schrödinger's original equation

$$\left[-\frac{\hbar^2}{2\mu}\vec{\nabla}^2 + V(\vec{x},t)\right]\Psi(\vec{x},t) = i\hbar\frac{\partial}{\partial t}\Psi(\vec{x},t)$$

is manifestly non-relativistic. Relativity treats space & time symmetrically, so an equation that is first-order in $\frac{\partial}{\partial t}$ but second-order in $\frac{\partial}{\partial \vec{x}} = \vec{\nabla}$ will never qualify.

Of course the more general form

$$\hat{H} |\psi\rangle = i\, \partial_t |\psi\rangle$$

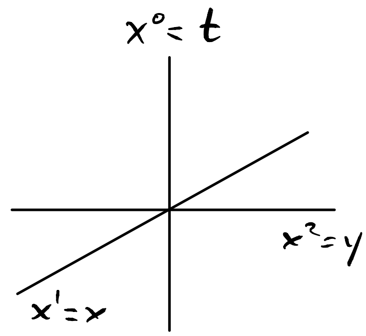
can be relativistic, but only if we choose the proper Hamiltonian $\hat{H}$. What should that be?

Schrödinger knew all this ("Schrödinger was no dummy" – Sidney Coleman), so he tried to develop a relativistic equation first. So we take a side trip to study Classical (Relativistic) Field Theory.

Let's introduce some relativistic notation.

Coordinates on spacetime are

$$x^\mu = \begin{pmatrix} x^0 \\ x^i \end{pmatrix} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$



where from now on we'll set $c = 1$, so $x^0 = t$, $x^1 = x$, etc. Spacetime has a geometry defined by a metric tensor

$$\eta_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

[Note that general-relativity people usually use the opposite sign convention, $\eta = (-+++)$. Particle physics & field theory use $+---$.]

The metric allows us to measure distances, or more properly the spacetime interval.

Given two points in spacetime x_1^μ and x_2^μ, with coordinate separation $\Delta x^\mu = x_2^\mu - x_1^\mu$, the spacetime interval between them is

$$
\begin{aligned}
(\Delta s)^2 &= \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu \\
&= \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu \\
&= -(\Delta t)^2 + (\Delta \vec{x})^2 .
\end{aligned}
$$

(Whether indices are "upper" or "lower" is crucially important.)

Here we're using the (Einstein) summation convention, where an index that appears twice (once up, once down) is summed over.

Note "conservation of indices":

- A summed-over index is called a "dummy," and appears exactly once upstairs and once downstairs.
- Free (non-dummy) indices must match in every term of an equation.

So things like

$$A_\alpha = B_\beta$$

or

$$X_{\mu\nu\mu} = Y_\nu$$

are just mistakes.

The quantity $(\Delta s)^2$, notice, can be positive, negative, or zero. Consider:

$$\Delta x^\mu_A = (0,1,0,0) \rightarrow \Delta s^2 = -1$$

$$\Delta x^\mu_B = (1,1,0,0) \rightarrow \Delta s^2 = 0$$

$$\Delta x^\mu_C = (1,0,0,0) \rightarrow \Delta s^2 = +1$$

We therefore define

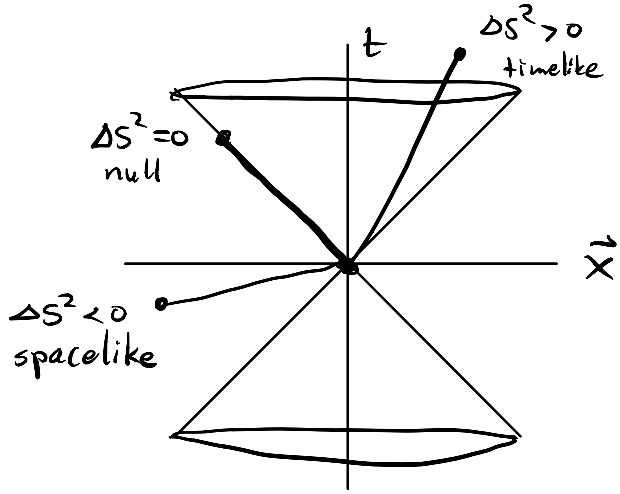
$\Delta s^2 < 0 \quad \leftrightarrow$ "spacelike"

$\Delta s^2 = 0 \quad \leftrightarrow$ "lightlike" or "null"

$\Delta s^2 > 0 \quad \leftrightarrow$ "timelike".

If $\Delta s^2 < 0$, the spatial distance is simply $\Delta r = \sqrt{-\Delta s^2}$. If $\Delta s^2 > 0$, the elapsed ("proper") time is $\Delta \tau = \sqrt{\Delta s^2}$.

Given any one point in spacetime, null rays (on which actual light rays move) radiate outward to define <u>light cones</u>.



The metric $\eta_{\mu\nu}$ is an example of a <u>tensor</u>. For our present low-tech purposes a tensor is just an indexed collection of functions that transforms in a certain way when we change coordinates:

$$x^\mu \to x^{\mu'} \implies \eta_{\mu\nu} \to \eta_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \eta_{\mu\nu}.$$

One favorite coordinate transformation is the <u>Lorentz transformation</u>. E.g. a <u>boost</u> in the x-direction with velocity v and boost factor $\gamma = 1/\sqrt{1-v^2}$ is given by

$$t' = \gamma t - \gamma v x, \qquad x' = -\gamma v t + \gamma x,$$

which implies

$$\frac{\partial x^\mu}{\partial x^{\mu'}} = \begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

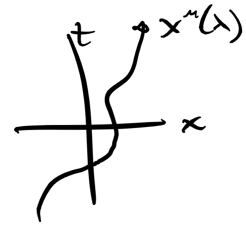
(as a matrix, the inverse of $\partial x^{\mu'}/\partial x^\mu$.)

We can imagine more elaborate coordinate transformations like Cartesian $\to$ polar, but won't need to here.

Check at home: the components of $\eta_{\mu\nu}$ are <u>unchanged</u> under Lorentz transformations. (Not true for almost all tensors.)

A few other tensors we will need.

Consider a particle on a trajectory $x^\mu(\lambda)$ through spacetime, where λ is some parameter. If it's a physical particle it will be moving on a timelike trajectory (slower than light). The infinitesimal proper time measured by a clock on this trajectory satisfies



$$d\tau^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu .$$

which implies

$$\tau(\lambda) = \int d\tau = \int \frac{d\tau}{d\lambda}\, d\lambda = \int \left(\eta_{\mu\nu} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{1/2} d\lambda .$$

So we can always use the proper time τ as a parameter along timelike trajectories, $x^\mu(\lambda) = x^\mu(\tau(\lambda))$.

This lets us define the <u>four-velocity</u> of a path at some point:

$$U^\mu = \frac{dx^\mu}{d\tau} \, . \qquad \text{(note upper index)}$$

The four-velocity is automatically normalized:

$$\eta_{\mu\nu} U^\mu U^\nu = +1$$

Check:

$$\eta_{\mu\nu} U^\mu U^\nu = \eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$$

$$= \frac{(\eta_{\mu\nu} dx^\mu dx^\nu)}{(d\tau)^2} = +1 .$$

The four-velocity naturally has an upper index, because the coordinate x^μ does (by convention, and all index placements follow from that).