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125c, Lecture Eighteen : 6/5/17

Continuing Classical Field Theory

Another useful tensor is the gradient of a function. (Single functions of spacetime are called "scalar fields.")

Gradient: $\partial_\mu \phi \equiv \dfrac{\partial \phi}{\partial x^\mu}$.

The gradient has a lower index — on the RHS we see x^μ, but its in the "denominator," so it corresponds to a lower index.

Note that partial derivatives commute, so

$$\partial_\mu \partial_\nu \phi = \partial_\nu \partial_\mu \phi .$$

We can use the metric to "lower indices."
For example: $U_\mu \equiv \eta_{\mu\nu} U^\nu$.
Note μ is a "free" index, while the dummy index ν is summed over.

It would be nice to be able to also "raise indices." We do that with the <u>inverse metric</u> $\eta^{\mu\nu}$, defined by

$$\eta^{\mu\nu}\eta_{\nu\sigma} = \delta^\mu{}_\sigma \quad \text{(Kronecker delta)}.$$

In components,
$$\eta^{\mu\nu} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

Same as $\eta_{\mu\nu}$! But we care about upper vs lower indices so we'll maintain the distinction.

Now we can finally return to field theory. Remember the hope was to find a relativistic wave equation for QM. So we have some complex field $\Phi(\vec{x},t)$ that we hope will be like the wave function $\Psi(\vec{x},t)$ (but our hopes will be dashed). Let's make an equation for it that is:

- linear in Φ
- invariant under Lorentz transformations
- Made from tensors such as $\eta_{\mu\nu}$, $\eta^{\mu\nu}$, ∂_μ.

The beauty of tensors is that inventing Lorentz-invariant equations is trivial: just use only tensors, and make sure every term has the same index structure (same free upper & lower indices).

Since we want a linear equation, we can always write a constant times the field:

$$m^2 \Phi .$$

(Constant labelled "m^2" for reasons that will become evident later.)

We can also take the gradient:

$$\partial_\mu \Phi .$$

Sadly index structures don't match. But we can take <u>two</u> derivatives and "soak up the indices" with the inverse metric:

$$\eta^{\mu\nu} \partial_\mu \partial_\nu \Phi \equiv \partial_t^2 \Phi - \vec{\nabla}^2 \Phi .$$

Two dummy indices, but no free indices. This operator is so useful that it has a name, the "d'Alembertian," and a symbol:

$$\Box \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu = \partial^\mu \partial_\mu .$$

Now we have two terms with the same index structure, so we can immediately write down a relativistically invariant equation:

$$\Box \Phi + m^2 \Phi = 0.$$

This is the Klein-Gordon equation. It was first derived by Schrödinger, who tried to use it as an equation for the quantum wave function.

This idea failed, for a number of reasons. One is that, in QM, the probability is $|\psi|^2 = \psi^* \psi$, so we require

$$\int dx \, |\psi|^2 = 1.$$

Using the non-rel. Schrödinger eq., it's easy to show that $\int |\psi|^2 dx$ is conserved over time.

But the KG equation does not conserve $\int |\Phi|^2 dx$ (as you can check). There is a quantity that is conserved under KG evolution:

$$Q = \int dx \left[\Phi^* \partial_t \Phi - \Phi \partial_t \Phi^* \right].$$

Unfortunately, this quantity is not manifestly positive! So it's not a good candidate for a conserved probability.

Therefore: the KG equation is not, in any sense whatsoever, a relativistic version of Schrödinger's equation for the quantum wave function.

However, we can use the KG equation for a completely separate purpose: as an equation of motion for a classical scalar field. It is this kind of field that we will "quantize" to get quantum field theory. In other words, rather than starting with a particle with position $\vec{x} \in \mathbb{R}^3$ and constructing a wave function

$$\psi(\vec{x}) : \mathbb{R}^3 \to \mathbb{C},$$

we start with a field $\Phi(\vec{x}) : \mathbb{R}^3 \to \mathbb{C}$ and construct a "wave functional"

$$\Psi[\Phi(\vec{x})] : \mathcal{F}_{\mathbb{C}}(\mathbb{R}^3) \to \mathbb{C},$$

where $\mathcal{F}_{\mathbb{C}}(\mathbb{R}^3)$ means "complex-valued functions of $\mathbb{R}^3$." I.e. $\Psi \in \mathcal{F}_{\mathbb{C}}\left(\mathcal{F}_{\mathbb{C}}(\mathbb{R}^3)\right)$.

wave functional ↑ ← field

But to reiterate: Φ is <u>not</u> a quantum wave function, it's a classical field. Indeed, the KG equation works perfectly well with real-valued functions $\phi : \mathbb{R}^3 \to \mathbb{R}$:

$$\Box \phi + m^2 \phi = 0.$$

However, in field theory it's usually most convenient to work with an <u>action</u> formalism rather than directly with equations of motion. Recall that the action for a classical non-relativistic particle $q(t)$ takes the form

$$S[q(t)] = \int dt \; L(q, \dot{q}),$$

where L is the <u>Lagrangian</u>,

$$L = K - V$$

$$\text{kinetic energy} - \text{potential energy}.$$

The equations of motion can then be obtained by "varying the action" to derive the Euler-Lagrange equations:

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0.$$

Classical field theory is similar, except that instead of q we have some set of fields Φ^a, and the Lagrangian can be expressed as an integral over space of a Lagrange density, which depends on the Φ^a and their spacetime derivatives:

$$L = \int d^3x\, \mathcal{L}(\Phi^a, \partial_\mu \Phi^a)$$

The action is then

$$S = \int dt\, L = \int d^4x\, \mathcal{L}(\Phi^a, \partial_\mu \Phi^a),$$

and the Euler-Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial \Phi^a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)} \right) = 0.$$

The great thing about the Lagrange density (usually just called "the Lagrangian") is that it is itself a scalar – it has no free indices. So we are highly constrained in making up Lagrangians. For a single real field ϕ, all we have to work with are ϕ itself and its gradient squared:

$$\mathcal{L} = \mathcal{L}(\phi, \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi).$$

Note that $\eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$, $\partial^\mu \phi \partial_\mu \phi$, and $(\partial_\mu \phi)^2$ all mean the same thing.

The simplest nontrivial example would be (presciently choosing some conventions)

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)$$

$$= \frac{1}{2}(\dot{\phi})^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - V(\phi).$$

Note the resemblance to "kinetic energy minus potential energy." There is a new term in the middle, the "gradient energy," representing energy in the spatial variation of the field.

The Euler-Lagrange equations for this Lagrangian are

$$0 = \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)$$

$$= -\frac{dV}{d\phi} - \partial_\mu\left(\eta^{\mu\nu}\partial_\nu\phi\right)$$

$$= -\eta^{\mu\nu}\partial_\mu\partial_\nu\phi - \frac{dV}{d\phi}.$$

If we choose

$$V(\phi) = \frac{1}{2} m^2 \phi^2,$$

this is precisely the Klein-Gordon equation!

$$\Box \phi + m^2 \phi = 0.$$

The equation for a "free" (no non-linear "interaction terms"), "massive" (when we quantize m will be the actual mass of the corresponding particles), classical scalar field.

A good toy model for field theory in general. The only known fundamental scalar field is the Higgs. Other classical fields we might want to quantize:

- Electromagnetic fields $\vec{E}, \vec{B}$.
- gravitational field.

Both involve subtleties. According to GR, gravity is the curvature of spacetime. That means we promote the fixed metric $\eta_{\mu\nu}$ to a spacetime-dependent field:

$$\eta_{\mu\nu} \longrightarrow g_{\mu\nu}(x^\sigma).$$

Figuring out how to do that in a coordinate-independent way leads pretty directly to classical GR. We don't have a full quantum theory of gravity yet, though the weak-field approximation is well-understood.

For electromagnetism, we need to package $\vec{E}$ & $\vec{B}$ into a tensor. There are 6 components, which is too many for one 4-vector and too few for two 4-vectors.

The solution is an antisymmetric 2-index tensor:

$$F_{\mu\nu} = -F_{\nu\mu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}.$$

→ the "Maxwell Field Strength Tensor."

You can check that this particular arrangement explains how $\vec{E}$ and $\vec{B}$ transform into each other under Lorentz transformations.

But there is an even more fundamental mechanism at work. $\vec{E}$ & $\vec{B}$ can be written in terms of a scalar potential Φ_e and a (3-) vector potential $\vec{A}$ as

$$\vec{E} = -\vec{\nabla}\Phi_e - \partial_t \vec{A}$$

$$\vec{B} = \vec{\nabla}\times\vec{A}.$$

Secretly these are part of a 4-vector gauge potential:

$$A_\mu = \begin{pmatrix} A_0 \\ \vec{A} \end{pmatrix} = \begin{pmatrix} \Phi_e \\ \vec{A} \end{pmatrix} .$$

Then the field strength is simplicity itself:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

It is invariant under gauge transformations:

$$A_\mu \to A_\mu + \partial_\mu \theta,$$

simply because partial derivatives commute:

$$F_{\mu\nu} \to F_{\mu\nu} + \partial_\mu \partial_\nu \theta - \partial_\nu \partial_\mu \theta = F_{\mu\nu} .$$

It is A_μ, not $F_{\mu\nu}$, that we treat as the fundamental field variable in the action principle. The Lagrangian for electromagnetism is:

$$\mathcal{L}_{EM} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + A_\mu J^\mu ,$$

raised indices!

where J^μ is the "current 4-vector." Charge conservation implies

$$\partial_\mu J^\mu = 0,$$

which guarantees that $\mathcal{L}_{EM}$ is gauge-invariant. (Check that you recover Maxwell's eqns.)

We can combine our scalar Klein-Gordon theory with electromagnetism. But to do that we have to start with a <u>complex</u> scalar field, $\Phi : \mathbb{R}^4 \to \mathbb{C}$.

Roughly: anything coupled to E&M must allow for both positive & negative charges; for that we need two (real) scalars, which it's convenient to combine into a single complex field.

The Lagrangian for our complex scalar Φ is

$$\mathcal{L} = \eta^{\mu\nu}(\partial_\mu \Phi^*)(\partial_\nu \Phi) - m^2 \Phi^* \Phi .$$

(No $\frac{1}{2}$ because we treat Φ and Φ^* as independent variables when varying to get equations of motion.)

Notice a symmetry:

$$\Phi \to e^{i\theta} \Phi , \qquad \theta \in [0, 2\pi).$$

That's a "global" symmetry (change the field globally everywhere by the fixed amount $e^{i\theta}$).

A much more impressive symmetry would be a <u>local</u> symmetry:

$$\Phi \to e^{i\theta(x)} \Phi , \qquad \theta : \mathbb{R}^4 \to [0, 2\pi).$$

But our Lagrangian isn't invariant under this transformation, because

$$\partial_\mu \Phi \to \partial_\mu \left(e^{i\theta(x)} \Phi\right) = e^{i\theta(x)}\left(\partial_\mu \Phi + i \Phi \partial_\mu \theta\right).$$

But that troublesome $\partial_\mu \theta$ reminds us of $A_\mu \to A_\mu + \partial_\mu \theta$. Those were different θ's, but what if we made them the same? I.e. every time we send $\Phi \to e^{i\theta(x)}\Phi$, we also send $A_\mu \to A_\mu + \partial_\mu \theta$. And we also replace the partial derivative $\partial_\mu \Phi$ with a covariant derivative

$$D_\mu \Phi = (\partial_\mu - i A_\mu)\Phi$$
$$= \partial_\mu \Phi - i A_\mu \Phi.$$

Then

$$D_\mu \Phi \to \partial_\mu\left(e^{i\theta(x)}\Phi\right) - i\left(A_\mu + \partial_\mu \theta\right)e^{i\theta(x)}\Phi$$
$$= e^{i\theta(x)}\left[\partial_\mu \Phi + \cancel{i\Phi\partial_\mu\theta} - i A_\mu \Phi - \cancel{i\Phi\partial_\mu\theta}\right]$$
$$= e^{i\theta(x)}\left(\partial_\mu \Phi - i A_\mu \Phi\right)$$
$$= e^{i\theta} D_\mu \Phi.$$

So we now have a Lagrangian

$$\mathcal{L} = \eta^{\mu\nu}(D_\mu \Phi)^* (D_\nu \Phi) - m^2 \Phi^* \Phi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}.$$

That is invariant under gauge transformations

$$\Phi \to e^{i\theta(x)} \Phi, \qquad A_\mu \to A_\mu + \partial_\mu \theta.$$

This can be generalized. Think of $e^{i\theta}$ as a 1×1 unitary matrix:

$$G = (e^{i\theta}), \qquad U^\dagger U = 1.$$

The set of all such matrices is called

$$U(1) = \text{unitary } 1\times 1 \text{ matrices.}$$

We can, of course, generalize:

$$U(n) = \text{unitary } n\times n \text{ matrices.}$$

But the determinant of a unitary matrix is just a phase, $e^{i\alpha}$. So we can decompose:

$$G = e^{i\alpha/n} \cdot \bar{G}$$

generic $n\times n$ unitary

phase

$n\times n$ unitary with determinant $|\bar{G}| = 1$.

I.e. we can write

$$\underset{\substack{\uparrow \\ n\times n \text{ unitary} \\ \text{matrices}}}{U(n)} = \underset{\substack{\uparrow \\ 1\times 1 \text{ unitary} \\ \text{matrices}}}{U(1)} \times \underset{\substack{\uparrow \\ n\times n \text{ unitary} \\ \text{matrices with} \\ \text{determinant } = 1.}}{SU(n)}$$

So without loss of generality we ask: in addition to $U(1)$ gauge theory, can we invent $SU(n)$ gauge theories?

Sure. Consider an n-component complex "vector":

$$\Phi^a = \begin{pmatrix} \Phi^1 \\ \Phi^2 \\ \vdots \\ \Phi^n \end{pmatrix}$$

(a vector in an abstract n-dim space we just made up, not in real space or space time.)

Then a "gauge transformation" is

$$\Phi^a \to \underbrace{G^a{}_b(x)}_{\in SU(n)} \Phi^b$$

Jargon: if gauge transformation matrices <u>commute</u>, as with $U(1)$, the symmetry is <u>abelian</u>; if they don't, it's <u>non-abelian</u>.

The covariant derivative is

$$(D_\mu \Phi)^a = \partial_\mu \Phi^a - i A_\mu{}^a{}_b \Phi^b$$

So now the gauge fields $A_\mu{}^a{}_b$ are matrices for each μ. In fact: traceless, Hermitian $n \times n$ matrices. (Technically: elements of the Lie algebra of the Lie group $SU(n)$.) Under gauge transformations they transform as

$$A_\mu{}^a{}_b \to G^a{}_c A_\mu{}^c{}_d (G^d{}_b)^* + G^a{}_c \partial_\mu (G^c{}_b)^* .$$

The field strength tensor is

$$\underbrace{F_{\mu\nu}{}^a{}_b}_{\substack{\text{traceless,} \\ \text{Hermitian} \\ \text{(for each } \mu\nu)}} = \partial_\mu A_\nu{}^a{}_b - \partial_\nu A_\mu{}^a{}_b + A_\mu{}^a{}_c A_\nu{}^c{}_b - A_\nu{}^a{}_c A_\mu{}^c{}_b .$$

Left as an exercise: under gauge transformations,

$$F_{\mu\nu}{}^a{}_b \to G^a{}_c F_{\mu\nu}{}^c{}_d (G^d{}_b)^* .$$

So here is the Lagrangian for a scalar field coupled to an SU(n) gauge theory:

$$\mathcal{L} = \eta^{\mu\nu} (D_\mu \Phi)^a (D_\nu \Phi^*)_b - m^2 \Phi^a (\Phi^*)_a - \frac{1}{4} F^{\mu\nu a}{}_b F_{\mu\nu}{}^b{}_a .$$

Pleasingly compact & elegant.

The Standard Model of Particle Physics is an $SU(3) \times SU(2) \times U(1)$ gauge theory. The most straightforward part of the theory is the SU(3) part, which is Quantum Chromodynamics (QCD). There are no scalar fields, but there are fermion fields, the quarks. These come in a triplet of fields, whimsically denoted "colors" red, green, blue:

$$q^a = \begin{pmatrix} q^1 \\ q^2 \\ q^3 \end{pmatrix} .$$

(1,2,3) = (red, blue, green)

The QCD gauge fields are called <u>gluons</u>.

How many are there? Well,

$$A_\mu{}^a{}_b \in \text{traceless, Hermitian, } 3\times 3 \text{ complex matrices}$$

So the number of independent fields is

$$\underbrace{(3\times 3)}_{3\times 3\text{ matrices}} \times \underbrace{2}_{\text{complex}} - \underbrace{(3\times 3)}_{\text{Hermitian}} - \underbrace{1}_{\text{Traceless}} = 8 \text{ gluons.}$$

[# of gauge bosons = dimensionality of SU(n).]

The $SU(2)\times U(1)$ bit is more subtle.

The subtlety comes from <u>spontaneous symmetry breaking</u> via the Higgs field.

This induces a mixing:

$$SU(2) \times U(1)$$

symmetry breaking mixes gauge fields

weak interactions: massive $W^\pm$, Z bosons

electromagnetism: massless photon

Note: for the scalar field Φ, the (mass)^2 is the coefficient of the Φ^2 term in the Lagrangian. This is a general feature: for any field X, the mass^2 is the coefficient of the X^2 term in the Lagrangian, if any. We see that gauge fields apparently must be massless, because the relevant term isn't gauge invariant:

$$A^\mu A_\mu \to (A^\mu + \partial^\mu \theta)(A_\mu + \partial_\mu \theta) \neq A^\mu A_\mu .$$

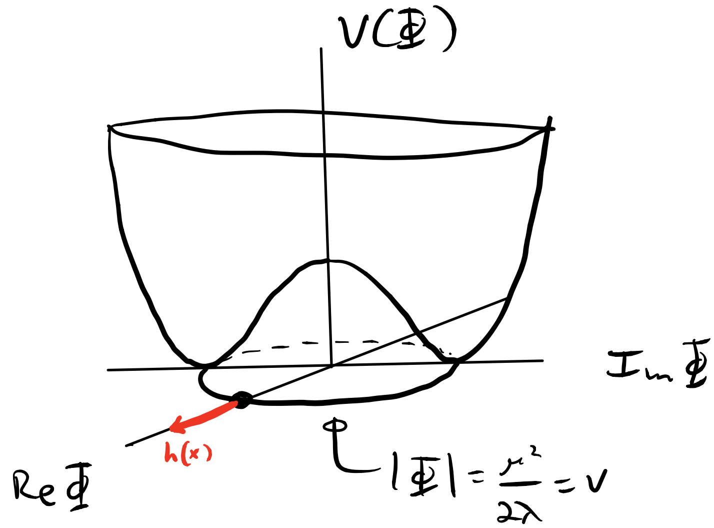
A mystery, since the W & Z bosons actually <u>are</u> massive.

Spontaneous symmetry breaking is the answer. Think of our original $U(1)$ scalar theory, but with a "Mexican Hat potential":

$$V(\Phi) = -\mu^2 |\Phi|^2 + \lambda |\Phi|^4 .$$

Now the minimum is not at $\Phi = 0$, but at

$$|\Phi| \equiv v = \frac{\mu^2}{2\lambda}.$$



In particle physics, we generally imagine that we are close to the vacuum (the minimum energy state of the theory), and field vibrations are small perturbations around that state. In the Mexican hat, the vacuum is at the minimum, $|\Phi| = v$. We can always do a gauge transformation $\Phi \to e^{i\theta}\Phi$ so that Φ is <u>real</u>, and write

$$\Phi(x) \longrightarrow v + h(x)$$

where $h(x)$ is a real scalar field.

Then watch what happens:

$$D_\mu \Phi = \partial_\mu \Phi - i A_\mu \Phi$$

$$\rightarrow \partial_\mu h - i v A_\mu - i h A_\mu$$

Then schematically (not being careful about indices):

$$|D_\mu \Phi|^2 \rightarrow \underbrace{(\partial_\mu h)^2}_{\substack{\text{kinetic term}\\ \text{for } h(x)}} + \underbrace{v^2 A^\mu A_\mu}_{\substack{\text{mass term}\\ \text{for } A_\mu}} + \underbrace{h^2 A^\mu A_\mu}_{\substack{\text{interaction}\\ \text{between } h \text{ \& } A_\mu}}.$$

Our complex scalar field has been replaced by a real scalar, and the gauge field A_μ now has a mass!

This is basically what happens in the Standard Model, except the broken symmetry is $SU(2) \times U(1)$, not just $U(1)$. The W & Z bosons become massive, the photon remains massless, and the leftover real scalar is the Higgs boson.