Sean Carroll, seancarroll@gmail.com

125c, Lecture Nineteen: 6/7/17

Quantum Field Theory (really)

Remember our real scalar field, with Lagrangian

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}m^2\phi^2,$$

obeying the Klein-Gordon equation

$$\Box\phi + m^2\phi = 0.$$

We'd like to quantize this. The first step is to convert from a Lagrangian to a Hamiltonian formulation. While the Lagrangian depends on ϕ and its derivatives, the Hamiltonian depends on $\phi(x)$ and a <u>conjugate momentum</u> $\pi(x)$. Let's recall how that works.

The Lagrange density is a scalar function on spacetime, and is therefore Lorentz invariant. A Hamiltonian, on the other hand, is not Lorentz invariant. It describes evolution forward in some specific time coordinate. (The underlying physics might very well be Lorentz invariant, but that symmetry is hidden in the Hamiltonian formulation.) To convert from $\mathcal{L}$ to the Hamiltonian density $\mathcal{H}$, we therefore separate out time- from space-derivatives:

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - \frac{1}{2}m^2\phi^2 .$$

The conjugate momentum is then

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}$$

Works out trivially in this case, but more complicated theories are more complicated.

The Hamiltonian then comes from the Lagrangian via a Legendre transform:

$$\mathcal{H}(\phi, \pi) = \pi\dot{\phi} - \mathcal{L}$$ (thought of as a function of ϕ & π, eliminating $\dot{\phi}$.)

$$= \pi\dot{\phi} - \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2$$

$$= \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 .$$

Note that it's non-negative, a good quality for an energy density to have.

The entire Hamiltonian is the <u>spatial</u> integral of this density:

$$H = \int d^3x \, \mathcal{H} .$$

The corresponding Hamilton equations of motion just recover $\dot{\phi} = \pi$ and the Klein-Gordon equation.

Consider a spatial Fourier transform:

$$\phi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3} e^{+i\vec{k}\cdot\vec{x}} \tilde{\phi}(\vec{k},t),$$

$$\pi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3} e^{+i\vec{k}\cdot\vec{x}} \pi(\vec{k},t).$$

What happens to the Hamiltonian? Consider just the π^2 term:

$$\frac{1}{2}\int d^3x\, \pi(\vec{x},t)^2$$

$$= \frac{1}{2(2\pi)^6}\int d^3x\, d^3k_1\, d^3k_2\, e^{i(\vec{k}_1+\vec{k}_2)\cdot\vec{x}}\, \tilde{\pi}(\vec{k}_1,t)\, \tilde{\pi}(\vec{k}_2,t).$$

But we know that

$$\int d^3\vec{x}\, e^{i(\vec{k}_1+\vec{k}_2)\cdot\vec{x}} = (2\pi)^3 \delta^{(3)}(\vec{k}_1+\vec{k}_2).$$

So we can do both $\int d^3x$ and $\int d^3k_2$, sending $\vec{k}_2 = -\vec{k}_1$:

$$\frac{1}{2}\int d^3x\, \pi^2 = \frac{1}{2}\int \frac{d^3k_1}{(2\pi)^3} \tilde{\pi}(\vec{k}_1,t)\, \tilde{\pi}(-\vec{k}_1,t).$$

But remember that $\pi(\vec{x},t)$ was <u>real</u>, which implies

$$\tilde{\pi}(-\vec{k}_1,t) = \tilde{\pi}^*(\vec{k}_1,t).$$

So

$$\frac{1}{2}\int d^3x\, \pi^2(x) = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\, |\tilde{\pi}(\vec{k},t)|^2.$$

Likewise for $\int d^3x\, \phi(x)^2$. For the gradient term we notice

$$\vec{\nabla}\phi(\vec{x},t) \to i\vec{k}\,\tilde{\Phi}(\vec{k},t).$$

So at last we have

$$\mathcal{H} = \int \frac{d^3k}{(2\pi)^3}\left[\frac{1}{2}|\tilde{\pi}(\vec{k},t)|^2 + \frac{1}{2}(\vec{k}^2+m^2)|\tilde{\Phi}(\vec{k},t)|^2\right].$$

For each $\vec{k}$, this is exactly the Hamiltonian of a simple harmonic oscillator of frequency

$$\omega(\vec{k}) = \sqrt{\vec{k}^2+m^2}.$$

A Klein-Gordon field is just a collection of oscillators <u>in Fourier space.</u>

So let's <u>quantize</u> these oscillators.

Recall the ordinary SHO:

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2$$

We promote $\hat{x}$ and $\hat{p}$ to operators, with canonical commutation relations

$$[\hat{x}, \hat{p}] = i$$

$\hbar = 1$.

The quantum Hamiltonian operator is

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2 \hat{x}^2.$$

It has eigenvalues

$$\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle, \qquad E_n = (n+\tfrac{1}{2})\omega$$

Interestingly, the ground-state energy isn't zero:

$$E_0 = \frac{1}{2}\omega.$$

"zero-point energy."

But this just represents a shift between the classical & quantum values of minimum energy; the overall value of the energy of an oscillator isn't meaningful, so we could just have analyzed

$$\hat{H}' = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{x}^2 - \frac{1}{2}\omega.$$

We can also study the SHO by introducing raising & lowering operators:

$$\text{lowering: } \hat{a} = \frac{1}{\sqrt{2\omega}}(\omega\hat{x} + i\hat{p}),$$

$$\text{raising: } \hat{a}^+ = \frac{1}{\sqrt{2\omega}}(\omega\hat{x} - i\hat{p}).$$

They have a commutator

$$[\hat{a}, \hat{a}^+] = 1.$$

The Hamiltonian becomes

$$\hat{H} = \frac{\omega}{2}(\hat{a}^+\hat{a} + \hat{a}\hat{a}^+)$$

$$= \omega(\hat{a}^+\hat{a} + \tfrac{1}{2}).$$

zero-point energy again.

This inspires us to define a "number operator":

$$\hat{n} \equiv \hat{a}^{+}\hat{a},$$

with eigenstates

$$\hat{n}\,|n\rangle = n\,|n\rangle$$

(the best equation in all of quantum mechanics).

The Hamiltonian is just

$$\hat{H} = \omega(\hat{n} + \tfrac{1}{2}).$$

The raising/lowering operators change the number eigenstate

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$\hat{a}^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle.$$

There is a unique ground ("vacuum") state that is annihilated by the lowering operator,

$$\hat{a}\,|0\rangle = 0.$$

Indeed, every number eigenstate can be found by starting with the vacuum and raising: (9)

$$|n\rangle = \frac{1}{\sqrt{n!}}\left(\hat{a}^\dagger\right)^n |0\rangle.$$

All of this carries over to field theory, with oscillators for each $\vec{k}$. (Or as we say, "for each Fourier mode.") Analogous to writing $(\hat{x},\hat{p}) \to (\hat{a}^\dagger, \hat{a})$, we can write

$$\hat{\phi}(\vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}\left(\hat{a}_{\vec{k}}\, e^{i\vec{k}\cdot\vec{x}} + \hat{a}^\dagger_{\vec{k}}\, e^{-i\vec{k}\cdot\vec{x}}\right)$$

$$\hat{\pi}(\vec{x}) = -i\int \frac{d^3k}{(2\pi)^3}\sqrt{\frac{\omega_{\vec{k}}}{2}}\left(\hat{a}_{\vec{k}}\, e^{i\vec{k}\cdot\vec{x}} - \hat{a}^\dagger_{\vec{k}}\, e^{-i\vec{k}\cdot\vec{x}}\right),$$

where $\omega_{\vec{k}} = (\vec{k}^2+m^2)^{1/2}$. Note that these are operators, not classical fields; they don't have time-dependence.

The operators $\hat{a}^\dagger_{\vec{k}}$, $\hat{a}_{\vec{k}}$ are creation/annihilation operators for each $\vec{k}$.

Their commutation relation is

$$[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}] = (2\pi)^3 \delta^{(3)}(\vec{k}-\vec{k}').$$

And the Hamiltonian becomes

$$\hat{H} = \int \frac{d^3k}{(2\pi)^3} \frac{\omega_{\vec{k}}}{2} \left(\hat{a}^\dagger_{\vec{k}} \hat{a}_{\vec{k}} + \hat{a}_{\vec{k}} \hat{a}^\dagger_{k} \right)$$

$$= \int \frac{d^3k}{(2\pi)^3} \omega_{\vec{k}} \left(\hat{a}^\dagger_{\vec{k}} \hat{a}_{\vec{k}} + \frac{1}{2} [\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k}}] \right).$$

The zero-point energy is there again, but now it's

$$\frac{1}{2} \int \frac{d^3k}{(2\pi)^3} [\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k}}] = \frac{1}{2} \int d^3k \, \delta^{(3)}(0)$$

$$= \infty.$$

This seems bad — an infinite energy density in the QFT vacuum. But again we can just subtract off $\frac{1}{2}\omega_k$ for each mode — "renormalization."

Of course we could also subtract off any other constant we like. In flat spacetime it doesn't matter, since the overall scale of energy is irrelevant — but for gravity it does matter, since gravity couples to all energy.

We can make the vacuum energy zero, but we could make it anything. This is the "Cosmological Constant Problem." We think we've measured it observationally, obtaining $\rho_{vac} \sim 10^{-8}$ ergs/cm^3. Nobody knows why it's that number.

As with the SHO, we have a vacuum state,

$$\hat{a}_{\vec{k}} |0\rangle = 0 \qquad \forall\, \vec{k},$$

and a number operator for each mode,

$$\hat{n}_{\vec{k}} = \hat{a}^{+}_{\vec{k}} \hat{a}_{\vec{k}} \ .$$

We can make a state with $n_{\vec{k}}$ excitations with exactly the same wave vector:

$$|n_{\vec{k}}\rangle = \frac{1}{\sqrt{n_{\vec{k}}!}} \left(\hat{a}^{+}_{\vec{k}}\right)^{n_{\vec{k}}} |0\rangle, \qquad \hat{n}_{\vec{k}} |n_{\vec{k}}\rangle = n_{\vec{k}} |n_{\vec{k}}\rangle,$$

but also states with fixed numbers of excitations at various wave numbers $\{\vec{k}_i\} = \{\vec{k}_1, \vec{k}_2 \cdots \vec{k}_I\}$:

$$|n_1, n_2, \cdots n_I\rangle = \frac{1}{\sqrt{n_1!\, n_2! \cdots n_I!}} \left(\hat{a}^{+}_{\vec{k}_1}\right)^{n_1} \left(\hat{a}^{+}_{\vec{k}_2}\right)^{n_2} \cdots \left(\hat{a}^{+}_{\vec{k}_I}\right)^{n_I} |0\rangle .$$

How do we interpret these states?

Subtracting off the zero-point energy, the Hamiltonian is

$$\hat{H} = \int \frac{d^3k}{(2\pi)^3} \, \omega_{\vec{k}} \, \hat{n}_k .$$

Then the vacuum satisfies

$$\hat{H} |0\rangle = 0.$$

The next-lowest energy would be a single excitation with $\vec{k} = \vec{k}_*$, i.e.

$$|1_{\vec{k}_*}\rangle = \hat{a}^{\dagger}_{\vec{k}_*} |0\rangle ,$$

for which

$$\hat{H} |1_{\vec{k}_*}\rangle = \int \frac{d^3k}{(2\pi)^3} \, \omega_{\vec{k}} \, \hat{n}_{\vec{k}} \, \hat{a}^{\dagger}_{\vec{k}_*} |0\rangle$$

$$= \int \frac{d^3k}{(2\pi)^3} \, \omega_{\vec{k}} \left(\hat{a}^{\dagger}_{\vec{k}} \, \hat{a}_{\vec{k}} \, \hat{a}^{\dagger}_{\vec{k}_*} \right) |0\rangle$$

Now use

$$\hat{a}_{\vec{k}} \hat{a}^{+}_{\vec{k}_*} |0\rangle = \left(\cancel{\hat{a}^{+}_{\vec{k}_*} \hat{a}_{\vec{k}}} + [\hat{a}_{\vec{k}}, \hat{a}^{+}_{\vec{k}_*}] \right) |0\rangle$$

$$= (2\pi)^3 \delta(\vec{k} - \vec{k}_*) |0\rangle$$

to obtain

$$\hat{H} |1_{\vec{k}_*}\rangle = \int \frac{d^3k}{(2\pi)^3} \omega_{\vec{k}} (2\pi)^3 \delta(\vec{k} - \vec{k}_*) \hat{a}^{+}_{\vec{k}} |0\rangle$$

$$= \omega_{\vec{k}} |1_{\vec{k}_*}\rangle .$$

So the one-particle state of definite momentum is an energy eigenstate, with energy

$$E = \omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2} .$$

This is precisely the energy of a single particle, with mass m and momentum $\vec{k}_*$. (That's why we used "m" in the KG equation way back.)

Same story holds for multi-excitation states. Therefore: we interpret energy eigenstates

$$\hat{H}\,|n_1, n_2 \cdots n_I\rangle = \Big(\sum_{\{n_i\}} \omega_{\vec{k}_i}\Big)\,|n_1, n_2 \cdots n_I\rangle$$

as states containing $N = \sum_i n_i$ particles.

(These particles are plane waves, but it's straightforward to combine states with different $\vec{k}$'s to make wave packets.)

They are identical particles, and it's not hard to show they obey Bose-Einstein statistics - they're bosons.

The set of all states of the form

$$|n_1 \cdots n_I\rangle \quad \text{for } \{\vec{k}_i\}, \; i = 1 \cdots I$$

form a basis, called the Fock basis.

Sometimes we call the associated Hilbert space "Fock space," but it's just Hilbert space.

We've shown that the Lagrangian for a real scalar field given by

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}m^2\phi^2$$

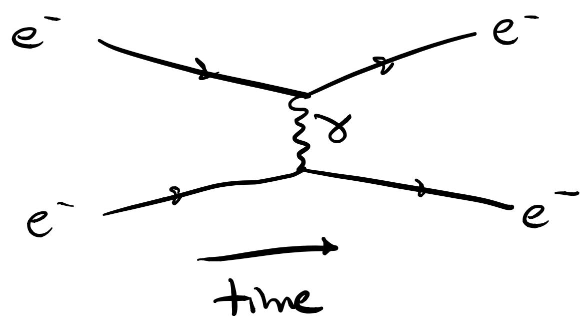
can be thought of as a theory of boson particles with mass m. You can also show that they are spin-0 bosons. Furthermore, they are <u>free</u> — the particles don't interact with each other. This stems from the fact that the KGE $(\Box + m^2)\phi = 0$ is linear in ϕ.

We can add non-linearities by adding terms to the potential, e.g.

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \gamma\phi^3 + \lambda\phi^4 + \ldots$$

Then it's impossible to solve exactly, and we generally turn to perturbation theory.

In QFT, perturbation theory generally means Feynman Diagrams. These diagrams are a way to calculate the scattering amplitudes for various processes. E.g. the scattering of two electrons via photon exchange includes a term

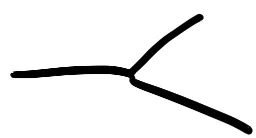


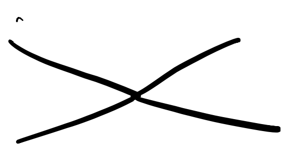
Each diagram corresponds to a number, and we add contributions from every possible number to get an amplitude, which we then square to get a probability. Happily, more complicated diagrams often (not always) give a smaller contribution, so we only have to compute simple ones.

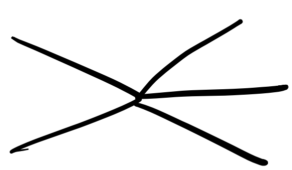
There is a simple relation between terms in the Lagrangian and the basic pieces from which Feynman diagrams are constructed. Quadratic terms in $\mathcal{L}$ give the propagator, which simply moves a particle from one spacetime point to another:

$$\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + \frac{1}{2}m^2\phi^2 \longleftrightarrow$$ 

Higher-order terms correspond to vertices, with one line for each field operator in that term:

$$\gamma\phi^3 \longleftrightarrow$$ 

$$\lambda\phi^4 \longleftrightarrow$$ 

$$\epsilon\phi^5 \longleftrightarrow$$ 

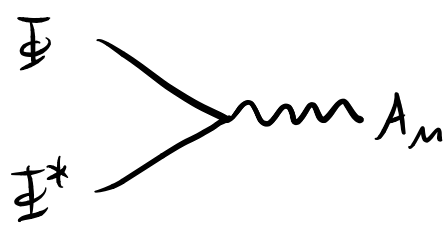
etc.

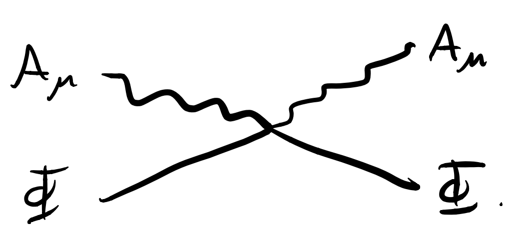
If we think back to our gauge theory, replacing partial derivatives with covariant derivatives induced an interaction between our complex scalar Φ (with propagator $\longrightarrow$) and the photon A_μ (with propagator $\sim\sim\sim$):

$$|D_\mu \Phi|^2 = (\partial_\mu \Phi^* + i A_\mu \Phi^*)(\partial^\mu \Phi - i A^\mu \Phi)$$

$$= |\partial_\mu \Phi|^2 + 2i A^\mu \, \mathrm{Im}(\Phi \partial_\mu \Phi^*) + A_\mu A^\mu |\Phi|^2 .$$

Second and third terms represent interactions:





In this way, all interactions between matter and photons (and gluons, etc.) are fixed by the requirement of gauge invariance.