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125c, Lecture Two : 4/5/16

Two Qubits!

$$\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$$

Qubit A: $\{ |0\rangle_A, |1\rangle_A \}$

Qubit B: $\{ |0\rangle_B, |1\rangle_B \}$

System: $|\Psi\rangle = \sum_{ij} \alpha_{ij} |i\rangle_A |j\rangle_B$

$$= \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle .$$

State might look like

$$|\Psi\rangle = (\alpha|0\rangle_A + \beta|0\rangle_A) \otimes (\gamma|0\rangle_B + \delta|1\rangle_B)$$

"Factorizable" or "Tensor Product State."

But maybe not, e.g.

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B \right)$$

$$\equiv \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle \right)$$

Entangled!

$|\psi\rangle$ is entangled if it <u>can't</u> be written as a single tensor product, $|\psi\rangle_A \otimes |\psi\rangle_B$.

Basis

Let $|00\rangle = |0\rangle_A \otimes |0\rangle_B$, etc.

Then a basis for the 2-qubit system is

$$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}.$$

John Bell popularized entangled Bell States:

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle \pm |11\rangle\right)$$

$$|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle \pm |10\rangle\right)$$

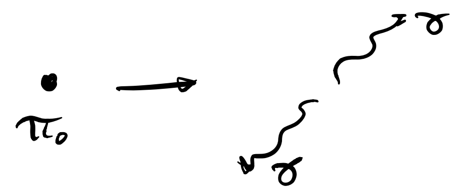
Bell states form a basis:

$$\{|\Phi^+\rangle, |\Phi^-\rangle, |\Psi^+\rangle, |\Psi^-\rangle\}.$$

Entanglement is everywhere.

E.g. pion decay to two photons:

$$\pi^0 \to 2\gamma$$



Polarizations are anti-correlated:

$$|\pi_0\rangle \to \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$$

where $|0\rangle = RH$, $|1\rangle = LH$.

Entanglement depends on factorization of Hilbert Space.

Consider: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

$\mathcal{H}$: $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ $\qquad$ $\mathcal{H}_A$: $\{|0\rangle_A, |1\rangle_A\}$ $\qquad$ $\mathcal{H}_B$: $\{|0\rangle_B, |1\rangle_B\}$

[Equals sign here really means "vector space isomorphism."]

We can also write $\mathcal{H} = \mathcal{H}_C \otimes \mathcal{H}_D$

$\mathcal{H}$: $\{|22\rangle, |23\rangle, |32\rangle, |33\rangle\}$ $\qquad$ $\mathcal{H}_C$: $\{|2\rangle_C, |3\rangle_C\}$ $\qquad$ $\mathcal{H}_D$: $\{|2\rangle_D, |3\rangle_D\}$

where

$$|22\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|23\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$|32\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

$$|33\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

Then $|\Psi\rangle = |22\rangle$ is entangled in $\mathcal{H}_A \otimes \mathcal{H}_B$, but a tensor product (unentangled) in $\mathcal{H}_C \otimes \mathcal{H}_D$.

Spooky Action

Consider two spins, $|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$.

Send spin B very far away.

Now let's measure our spins, simultaneously.

Probability of measuring $\hat{X}$ and obtaining $|x_0\rangle$ is

is $P_{\Phi}(x_0) = \langle \Phi | \hat{\Pi}_{x_0} | \Phi \rangle = \langle \Phi | x_0 \rangle\langle x_0 | \Phi \rangle$

$= |\langle x_0 | \Phi \rangle|^2$.

For us: $\text{Prob.}(|ij\rangle) = p(i,j) = |\langle ij | \Phi^+ \rangle|^2$

$i,j \in \{0,1\}$

I.e. $p(00) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$

$p(11) = 1/2$

$p(01) = p(10) = 0$.

<u>Unconditional</u> probability for Alice:

$$P_A(i) = \sum_j p(i,j) = p(i,0) + p(i,1)$$

So $P_A(0) = 1/2$, $P_A(1) = 1/2$.

What if only Alice measures?

Say outcome is $|0\rangle_A$.

I.e. we project using

$$\hat{\Pi}_{A,0} = \left(|0\rangle_A\langle 0|\right) \otimes \mathbb{1}_B .$$

$$P_A(0) = \langle \Phi^+ | \hat{\Pi}_{A,0} | \Phi^+ \rangle$$

Note $\hat{\Pi}_{A,0} | \Phi^+ \rangle = \frac{1}{\sqrt{2}} |00\rangle$.

$$\rightarrow P_A(0) = \langle \Phi^+ | \left(\tfrac{1}{\sqrt{2}} |00\rangle\right) = 1/2 \langle 00|00\rangle = 1/2 .$$

$$P_A(1) = 1/2 .$$

After measuring, state "collapses" to

$$|\Psi'\rangle = \hat{\Pi}_{A,0} | \Phi^+ \rangle = \frac{1}{\sqrt{2}} |00\rangle .$$

If we see A in $|0\rangle$,
we instantly know ~~B~~ B in $|0\rangle$.
(Likewise $A = |1\rangle$ implies $B = |1\rangle$, of course.)

Einstein, Podolsky & Rosen 1935 (EPR).

Does this bother you?

Should it?

Faster-than-light communication?

Before Alice measures, Bob's probabilities:

$$p_B(0) = 1/2, \quad p_B(1) = 1/2.$$

After Alice measures $|0\rangle_A$, Bob has probability 1 of measuring $|0\rangle_B$.

But Bob doesn't know Alice's outcome!

He only knows she had a 50/50 chance.

So to Bob, it's still $p_B(0) = p_B(1) = 1/2$.

Can Alice affect this?

Without measuring her qubit, she can do a unitary on it: $|\Psi\rangle \to (\hat{U}_A \otimes \mathbb{1}_B)|\Psi\rangle$

$$U_A : |0\rangle_A \to |u_0\rangle_A = \alpha|0\rangle + \beta|1\rangle,$$

$$|1\rangle_A \to |u_1\rangle_A = \gamma|0\rangle + \delta|1\rangle.$$

$$|\Psi'\rangle = \hat{U}_A \otimes \mathbb{1}_B |\Phi^+\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|u_0\rangle_A|0\rangle_B + |u_1\rangle_A|1\rangle_B\right).$$

Bob doesn't notice – his probability is still $p_B'(0) = p_B'(1) = 1/2$.

What if Alice now measures in the <u>original</u> basis?

Joint probabilities again:

$$p'(i,j) = |\langle ij|\Psi'\rangle|^2$$

$$\rightarrow \left.\begin{aligned} p'(0,0) &= \tfrac{1}{2}\,|\langle 0|u_0\rangle|^2 \\ p'(0,1) &= \tfrac{1}{2}\,|\langle 0|u_1\rangle|^2 \\ p'(1,0) &= \tfrac{1}{2}\,|\langle 1|u_0\rangle|^2 \\ p'(1,1) &= \tfrac{1}{2}\,|\langle 1|u_1\rangle|^2 \end{aligned}\right\} \begin{aligned} &p'(i,j) \\ &= \tfrac{1}{2}|\langle i|u_j\rangle|^2 \end{aligned}$$

Total probability for Bob:

$$P_B'(j) = \sum p'(i,j) = \frac{1}{2}\sum_i |\langle i|u_j\rangle|^2$$

$$= \frac{1}{2}\sum_i \langle u_j|i\rangle\langle i|u_j\rangle$$

$$= \frac{1}{2}\langle u_j|u_j\rangle = 1/2. \quad \checkmark$$

Bob's probabilities are unchanged by Alice's shenanigans.

→ "No-communication theorem"
(via entanglement)

Does that make you feel better?

Density Operators (Matrices)

All those conditional probabilities were a bit clunky. Would be nicer to talk directly about subsystems, even if entangled.

Problem: info about such subsystems is incomplete, if we don't know the whole state.

Solution: consider mixed states. A probabilistic ensemble of wave functions.

Note: can't just add! $\alpha|\psi_1\rangle + \beta|\psi_2\rangle$ isn't a statistical mixture, it's just a new state.

To any state $|\psi\rangle$, assign a density operator:

$$\hat{\rho} = |\psi\rangle\langle\psi| \qquad (\hat{\rho} \in L^2(\mathcal{H}))$$

For a mixture of states $|\psi_i\rangle$, each with probability p_i, we write

$$\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i| \,, \qquad p_i \in \mathbb{R}^+, \quad \sum_i p_i = 1.$$

Pure states: given $\hat{\rho}$, $\exists\, |\psi\rangle$ s.t. $\hat{\rho} = |\psi\rangle\langle\psi|$.

Mixed states (which most $\hat{\rho}$'s are): no such $|\psi\rangle$.

Properties of density operators:

- Hermitian, $\boxed{\hat{\rho}^{\dagger} = \hat{\rho}.}$

  check: $\hat{\rho}^{\dagger} = \left(\sum_i p_i |\psi\rangle\langle\psi|\right)^{\dagger} = \sum_i p_i |\psi_i\rangle\langle\psi_i|.$

- Unit trace: $\boxed{\mathrm{Tr}\,\hat{\rho} = 1.}$

  What's a trace?

  Choose a basis $|\phi_a\rangle$.

  Matrix elements: $\rho_{ab} \equiv \langle\phi_a|\hat{\rho}|\phi_b\rangle$

  Trace: $\mathrm{Tr}\,\hat{\rho} = \sum_a \rho_{aa} = \sum_a \langle\phi_a|\hat{\rho}|\phi_a\rangle.$

  So
$$\mathrm{Tr}\,\hat{\rho} = \sum_{a,i} p_i \langle\phi_a|\psi_i\rangle\langle\psi_i|\phi_a\rangle$$
$$= \sum_{a,i} p_i \langle\psi_i|\underbrace{\phi_a\rangle\langle\phi_a}_{\hookrightarrow\text{ complete set of states}}|\psi_i\rangle$$
$$= \sum_i p_i \langle\psi_i|\psi_i\rangle$$
$$= \sum_i p_i = 1.$$