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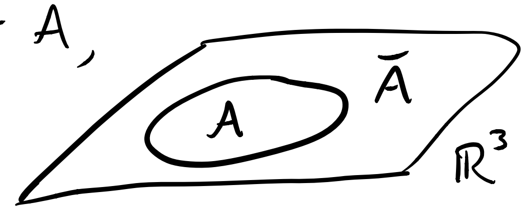
125c, Lecture Twenty : 6/9/17

The vacuum state $|0\rangle$ in a QFT has a high degree of spatial entanglement. That is, if we divide space into some region A and its complement $\bar{A}$,



we can find a basis (different from the plane waves above) of modes exclusively inside or outside of A. I.e. Hilbert space factorizes:

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}},$$

and our claim is that the parts of $|0\rangle$ in $\mathcal{H}_A$ and $\mathcal{H}_{\bar{A}}$ are highly entangled with each other (even in the vacuum!).

[For some theories, e.g. gauge theories, there are subtle obstructions to this factorization.]

Evidence: the Reeh-Schlieder Theorem.

Consider the set of all operators $\{\mathcal{O}_a^{(A)}\}$ that act only on $\mathcal{H}_A$ (and therefore commute with operators acting on $\mathcal{H}_{\bar{A}}$).

Then the set of states of the form

$$\mathcal{O}_a^{(A)}|0\rangle$$

is dense in $\mathcal{H}$. I.e., any state in Hilbert space can be approximated arbitrarily well by a state of this form.

Informally: by acting on the vacuum in my living room, I can make the Taj Mahal appear on the Moon.

This surprising claim is ultimately a result of entanglement.

Toy example: a Bell pair of qubits:

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$$

Claim: we can make any state in Hilbert space by acting just on the first qubit.

Proof: by construction. Consider these states obtained by acting on the 1st qubit:

$$|\alpha\rangle = (\hat{\sigma}_z \otimes \mathbb{1})|\Psi^+\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

$$|\beta\rangle = (\hat{\sigma}_x |0\rangle\langle 0| \otimes \mathbb{1})|\Psi^+\rangle = \frac{1}{\sqrt{2}}|11\rangle$$

$$|\gamma\rangle = (\hat{\sigma}_x |1\rangle\langle 1| \otimes \mathbb{1})|\Psi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle$$

$|\alpha\rangle$ is the Bell state $|\Psi^-\rangle$. And the other two Bell states are

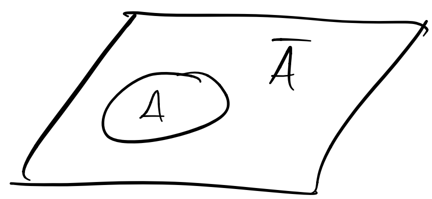
$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle \pm |11\rangle\right) = |\gamma\rangle \pm |\beta\rangle.$$

Bell states are a basis. Therefore, we can make any state of two qubits by some linear combination of the operators above.

Reeh-Schlieder is the same basic logic, though the proof is quite involved.

It illustrates the fact that the QFT ground state is highly entangled.

Another manifestation: calculate the reduced density matrix



$$\hat{\rho}_A = \mathrm{Tr}_{\bar{A}} |0\rangle\langle 0| .$$

The von Neumann entropy diverges:

$$S_A = -\mathrm{Tr}\, \hat{\rho}_A \log \hat{\rho}_A = \infty .$$

There are an infinite # of degrees of freedom, and they are entangled across the boundary. (At least in QFT. Quantum gravity suggests that entropy is finite, so maybe QFT breaks down when gravity gets involved.)

Yet another manifestation of entanglement:

The Unruh Effect. (a rough sketch.)

The lesson of relativity is that things once thought to be absolute (space, time) are relative to an observer's perspective. In QFT, that extends to the idea of "I am in a state with a certain number of particles." The Unruh effect says that a uniformly accelerating observer will detect particles even in (what an inertial observer would call) the vacuum.

Remember how we found particles in our scalar field theory: taking the spatial Fourier transform, and noticing that the Hamiltonian for ϕ was a sum of simple harmonic oscillators, one for each wave vector $\vec{k}$.

We wrote

$$\hat{\phi}(\vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{k}}}} \left(\hat{a}_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} + \hat{a}^{+}_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}} \right)$$

We didn't <u>have</u> to write $\hat{\phi}(\vec{x})$ this way. We could alternatively write

$$\hat{\phi}(\vec{x}) = \sum_i \frac{1}{\sqrt{2\omega_i}} \left[\hat{b}_i g_i(\vec{x}) + \hat{b}^{+}_i g^{*}_i(\vec{x}) \right],$$

where $g_i(\vec{x})$ is a different set of basis functions. (We've written it as a discrete sum, but it could really be an integral.) Then we could go through the same procedure, fixing commutation relations for the creation/annihilation operators $\hat{b}^{+}_i / \hat{b}_i$, finding vacuum and n-particle states, etc.

Here's the thing: the vacuum and n-particle states in the $\hat{a}$ expansion need not agree with those in the $\hat{b}$ expansion. In the "b-vacuum",

$$\hat{b}_i |0\rangle_b = 0,$$

we might have

$${}_b\langle 0| \hat{n}(a) |0\rangle_b \neq 0,$$

and vice-versa.

So how do we decide what's the "right" expansion to use? Surely there is some physical notion of a "particle detector" that either does or does not detect particles.

The answer is that an observer defines particles as "positive-frequency" oscillations of the field. I.e. we look for mode functions that can be completed to solutions of the KG equation of the form

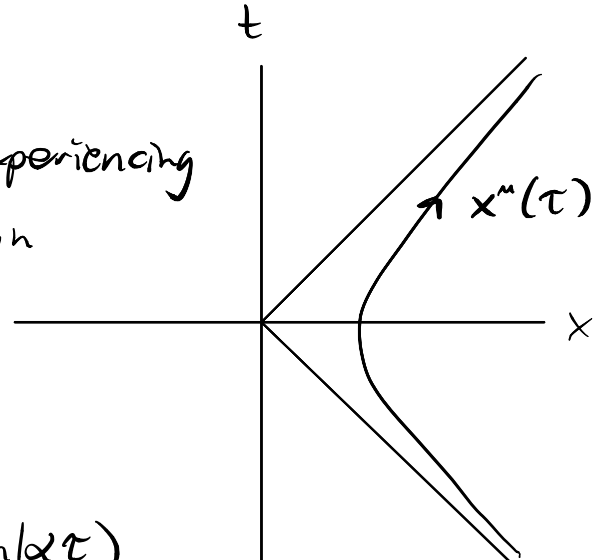
$$e^{-i\omega t} f_i(\vec{x}),$$

and associate them with annihilation operators. Corresponding "negative-frequency" solutions, associated with particle-creation operators, are of the form

$$e^{+i\omega t} f_i^*(\vec{x}).$$

Confusingly, the parameter ω is positive even for negative-frequency modes. What matters is $e^{-i\omega t}$ (positive-frequency) vs. $e^{+i\omega t}$ (negative-frequency).

So does that fix the right answer once and for all? Not exactly – because in relativity, observers measure time differently. What matters is the <u>proper</u> time along a path. The proper time for all <u>inertial</u> observers is proportional, so they agree on the vacuum and n-particle states. But non-inertial (accelerated) observers see things differently.



Consider an observer experiencing <u>constant</u> acceleration in the +x direction. Their world line will look like

$$x^0(\tau) = t(\tau) = \frac{1}{\alpha}\sinh(\alpha\tau)$$

$$x^1(\tau) = x(\tau) = \frac{1}{\alpha}\cosh(\alpha\tau). \qquad \text{(ignore } y \,\&\, z.)$$

We can actually choose new coordinates (ξ, η) adapted to this situation. Let

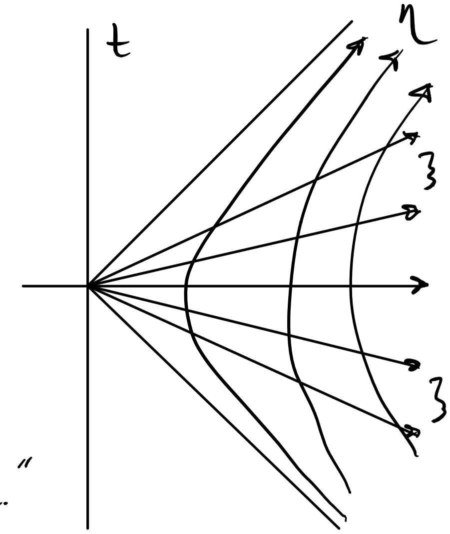
$$t = \frac{1}{a} e^{a\xi} \sinh(a\eta), \quad x = \frac{1}{a} e^{a\xi} \cosh(a\eta),$$

where the constant "a" is just a choice of scale, which we'll set $= \alpha$.

These provide good coordinates in the region

$$x \geq |t|,$$

known as the "Rindler wedge."



Then the constant-acceleration path is

$$\eta(\tau) = \frac{\alpha}{a}\tau, \quad \xi(\tau) = \frac{1}{a}\ln\left(\frac{a}{\alpha}\right).$$

In particular, for $\alpha = a$ we simply have

$$\eta = \tau, \qquad \xi = 0.$$

For this observer, η is their proper time.

A long and unenlightening calculation shows that there exist "Rindler modes" $g_i^R(\xi)$ in terms of which we can expand

$$\hat{\phi}(\xi) = \sum_i \frac{1}{\sqrt{2\omega_i}} \left[\hat{b}_i \, g_i^R(\xi) + \hat{b}_i^+ \, g_i^{R*}(\xi) \right],$$

such that solutions to the KGE are

$$e^{-i\omega\eta} g_i^R(\xi), \qquad e^{+i\omega\eta} g_i^{R*}(\xi).$$

We can construct vacuum states etc, and indeed we find that there exist "Rindler particles" in the ordinary Minkowski vacuum. In fact, we get

$${}_M\langle 0 | \hat{b}_i^+ \hat{b}_i | 0 \rangle_M = \frac{1}{e^{2\pi\omega/a} - 1} V,$$

where V is a constant equal to the volume of space.

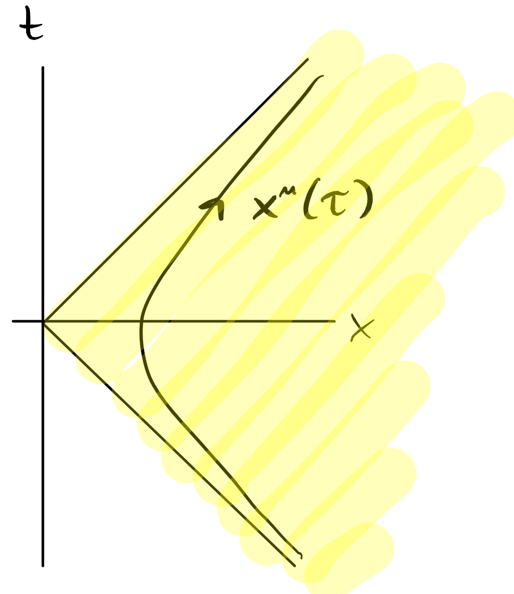
If we divide by V to get the number <u>density</u> of Rindler particles, what we're left with is a Planck (thermal) spectrum, with temperature

$$\boxed{T = \frac{a}{2\pi}}.$$

Thus, an accelerated observer in the vacuum observes a thermal bath of particles, called "Unruh Radiation."

There is a hand-waving explanation for this.

An observer moving with constant acceleration sees a <u>horizon</u> — nothing with $x < t$ can ever reach (or send signals to) them. It's as if that part of spacetime doesn't exist.

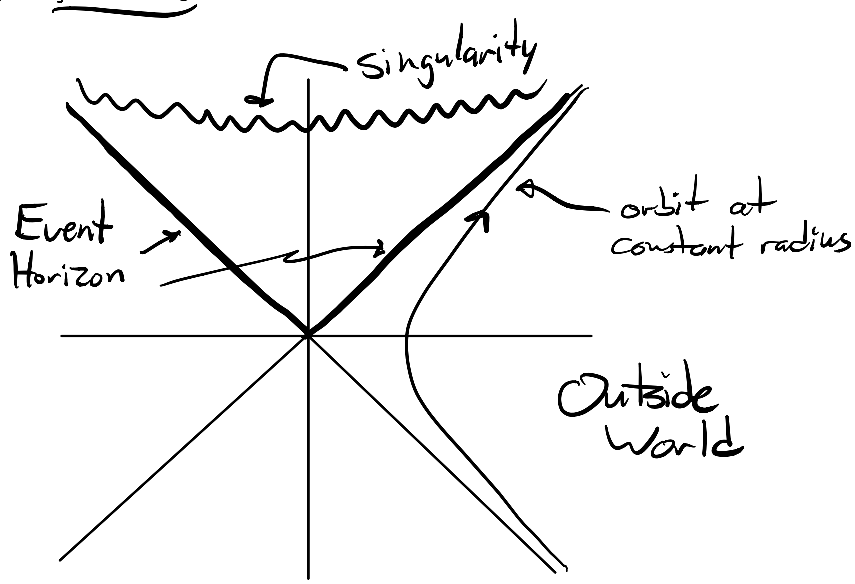


Therefore, effectively, the state they observe is a density matrix, obtained by tracing out degrees of freedom beyond their horizon. Because of entanglement, that state is highly mixed, even if the overall state (the Minkowski vacuum) is pure.

This explanation may or may not be helpful, but it's not crucial – what matters is the response of a detector, which is a completely local matter.

The Unruh effect is closely related to the Hawking effect – radiation from black holes. We don't have enough GR (or time) to do this justice, but the basic idea is straightforward.

The spacetime geometry of a Black Hole closely resembles that of the Rindler wedge. The BH horizon is actually <u>null</u>, and the singularity is <u>spacelike</u>. It's not located at the "middle" of the BH; it's located in the <u>future</u>.



Therefore, by analogous arguments to the Unruh effect, external observers see thermal radiation emitted from BH's, with temperature

$$T_H = \frac{1}{8\pi GM},$$

where M is the mass of the BH.

Just like an ordinary blackbody, the BH temperature is associated with an entropy:

$$S_{BH} = \frac{A}{4G} = 4\pi G M^2 = 10^{78}\left(\frac{M}{M_\odot}\right)^2,$$

where A is the area of the BH event horizon, and $M_\odot$ is the mass of the Sun. This is the Bekenstein-Hawking entropy, after the guy who proposed it (Jacob Bekenstein) and the guy who calculated it (Stephen Hawking).

The BH entropy is large. At the center of many spiral galaxies is a black hole with $M \geq 10^6 M_\odot$, so $S_{BH} \geq 10^{90}$.

Contrast this with the entropy of all the particles in the observable universe, which is only $S_{\text{particles}} \approx 10^{88}$.

Entropy increases. Given black-hole entropy plus regular thermodynamic entropy, we have the Generalized Second Law:

$$\Delta\left(S_{\text{thermo}} + \sum_{\text{BH's}} \frac{A_i}{4G}\right) \geq 0.$$

So in the process

$$\text{stuff} \longrightarrow \text{BH} \longrightarrow \text{Hawking radiation},$$

entropy increases at every step.