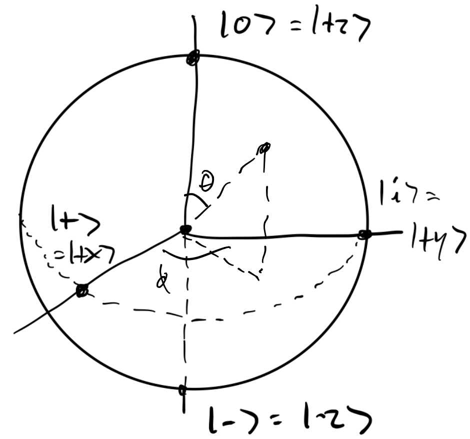
Sean Carroll, seancarroll@gmail.com

125c, Lecture Four : 4/12/16

Bloch vectors

Recall the Bloch sphere representation for qubits:



$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$
$$= \cos\tfrac{\theta}{2}|0\rangle + e^{i\phi}\sin\theta/2$$

We can extend this to (2×2) qubit density matrices $\hat{\rho}$.

Remember Pauli matrices:

$$\vec{\sigma} = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

Traceless & Hermitian. In fact, <u>any</u> 2×2 Hermitian matrix can be written

$$M = \underbrace{\vec{a}\cdot\vec{\sigma}}_{\text{traceless}} + \underbrace{b\,\mathbb{1}_{2\times 2}}_{\text{traceful}} = \begin{pmatrix} a_z + b & a_x - ia_y \\ a_x + ia_y & -a_z + b \end{pmatrix}$$

$\vec{a}, b \in \mathbb{R}$.

$$\mathrm{Tr}\, M = 2b.$$

②

This works for $\hat{\rho}$, where $\mathrm{Tr}\,\hat{\rho} = 1$.
So we can write any qubit density operator as

$$\hat{\rho} = \frac{1}{2}(\mathbb{1} + \vec{a}\cdot\vec{\sigma}), \quad \vec{a} = \text{"Bloch vector."}$$

<u>Ex.</u> $\vec{a} = +\hat{z} = (0, 0, 1)$

$$\rightarrow \hat{\rho} = \frac{1}{2}\left(\mathbb{1} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= |0\rangle\langle 0| = |+z\rangle\langle +z|$$

Also try: $\vec{a} = +\hat{x} = (1, 0, 0)$

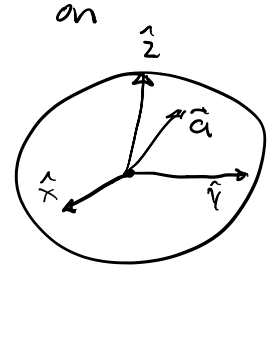
$$\hat{\rho} = \frac{1}{2}\left(\mathbb{1} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\right)$$

$$= \left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right]\left[\frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)\right]$$

$$= |+\rangle\langle +| = |+x\rangle\langle +x| \quad \checkmark$$

Bloch vectors "go where they should" on the Bloch sphere.



Problem Set:

$|\vec{a}| = 1 \longleftrightarrow \hat{\rho}$ is a pure state

$|\vec{a}| < 1 \longleftrightarrow \hat{\rho}$ is a mixed state

Time evolution

$$\hat{H}|\psi\rangle = i\partial_t |\psi\rangle \longrightarrow |\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle.$$

For a density operator, we therefore have

$$\hat{\rho}(\delta t) = \sum_i \underbrace{p_i}_{\text{doesn't change w/ time in a closed system}} |\psi_i(\delta t)\rangle\langle\psi_i(\delta t)|$$

$$= \sum_i p_i \hat{U}(\delta t)|\psi_i(0)\rangle\langle\psi_i(0)| U^+(\delta t)$$

$$= \hat{U}(\delta t)\,\hat{\rho}\,\hat{U}^+(\delta t)$$

$$= (1 - i\hat{H}\delta t + \cdots)\,\hat{\rho}\,(1 + i\hat{H}\delta t + \cdots)$$

$$\therefore \quad \delta\hat{\rho} = -i(\hat{H}\hat{\rho} - \hat{\rho}\hat{H})\,\delta t$$

$$\Longrightarrow \quad \frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}]$$ "von Neumann equation."

Evolution of a density matrix left by itself.

A reduced density matrix will generally be influenced by its environment — a trickier situation.

Entropy

Classically: imagine we have a space of states $\{x\} \in \Gamma$, and a distribution function $f(x)$, with $f: \Gamma \to \mathbb{R}^+$ and $\sum_x f(x) = 1$.

Then we define the entropy of $f(x)$ as

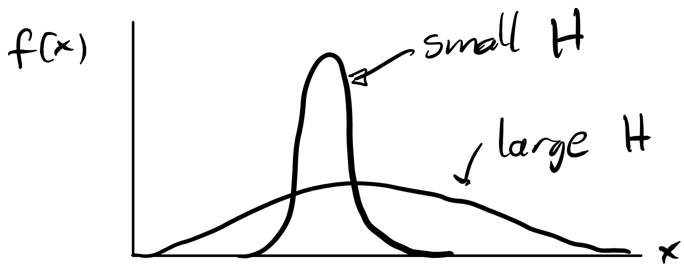
$$H[f] = -\sum_x f(x) \log f(x) \qquad [0 \cdot \log 0 = 0]$$

Many different definitions of entropy; this is the Gibbs (physics) or Shannon (info theory) entropy.

If there are N elements $x \in \Gamma$, then the maximum entropy is a uniform distribution, $f(x) = \frac{1}{N}$. $(\forall x)$

$$H[f] = -\sum_{x=1}^{N} \frac{1}{N} \log \frac{1}{N} = -\log \frac{1}{N} = \log N.$$

Higher entropy = less precise knowledge.



Entropy vs. "Information."

Closely related, but how?

Two senses of "information," with opposite relations to entropy!

Sense 1: (physics / statistical mechanics)

"Given $f(x)$, how much information do I have about x?"

Clearly, $f(x)$ is more "concentrated" when $H[f]$ is small — more information.

Sense 2 (communication / information theory):

Imagine "messages" as series of "events" x (letters / words / numbers), which appear with relative frequency $f(x)$.

So $f(x)$ might be the relative frequency of words in a text, or bits in Morse code, etc.

Now the question is: "How much information is conveyed in a typical message?"

6

The more concentrated $f(x)$ is, the <u>less</u> information you are likely to receive in a typical message!

$H[f]$ is small $\to$ <u>less</u> information

E.g. a coin flip, where $\{0,1\} = \{H, T\}$

$f(0)$ = probability of heads

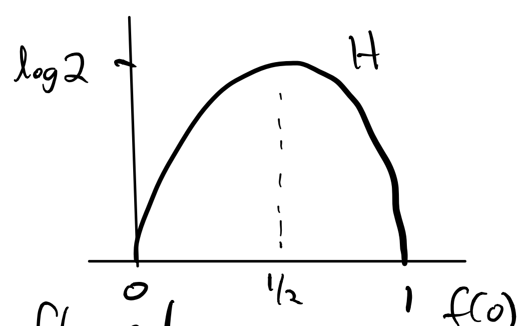
$f(1) = 1 - f(0)$ = probability of tails

Entropy: $H[f] = -\sum_x f(x) \log f(x)$

$$= -\left[f(0) \log f(0) + (1-f(0)) \log(1-f(0))\right]$$

e.g.

- $f(0) = 1/2$: $H = -\left(\frac{1}{2}(-\log 2) + \frac{1}{2}(-\log 2)\right) = \log 2.$
- $f(0) = 0$: $H = -\left(0 + (1-0)\log(1)\right) = 0.$



→ for a fair coin ($f(0) = 1/2$), you don't know what it will be ahead of time, so you gain information when it's flipped.

→ for an unfair coin ($f(0) = 0$), you learn nothing (it was always going to be tails).

Quantum (von Neumann) entropy.

$$S = -\mathrm{Tr}\, \hat{\rho} \log \hat{\rho}.$$

What the hell is log of an operator?

Well, $e^{\log x} = x$, and $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

So we can exponentiate operators:

$$e^{\hat{O}} = \mathbb{1} + \hat{O} + \frac{\hat{O}^2}{2!} + \frac{\hat{O}^3}{3!} + \cdots$$

Then

$\log \hat{\rho}$ = "the operator that, when exponentiated, gives $\hat{\rho}$."

Alternatively:

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$$

So we can define

$$\log \hat{\rho} = (\hat{\rho} - \mathbb{1}) - \frac{1}{2}(\hat{\rho} - \mathbb{1})^2 + \frac{1}{3}(\hat{\rho} - \mathbb{1})^3 - \cdots$$

When $\hat{\rho}$ is diagonal, $\rho_{ab} = \begin{pmatrix} p_1 & & & 0 \\ & p_2 & & \\ & & \ddots & \\ 0 & & & p_d \end{pmatrix}$,

then $(\log \hat{\rho})_{ab} = \begin{pmatrix} \log p_1 & & & 0 \\ & \log p_2 & & \\ & & \ddots & \\ 0 & & & \log p_d \end{pmatrix}$.

$(d = \dim \mathcal{H})$

So we can check

$$S[\hat{\rho}] = -\mathrm{Tr}\, \hat{\rho} \log \hat{\rho}$$

$$= -\sum_{i=1}^{d} p_i \log p_i \,.$$

→ matches Gibbs/Shannon.

E.g. $\hat{\rho} = |\psi\rangle\langle\psi|$ (pure).

Then $S[\hat{\rho}] = -1 \cdot \log(1) = -1 \cdot 0 = 0$.

→ pure-state density operators have zero entropy.

But: $\hat{\rho} = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1| \right)$

→ $S[\hat{\rho}] = -\left(\frac{1}{2} \log \frac{1}{2} + \frac{1}{2} \log \frac{1}{2} \right) = \log 2$.

→ mixed-state density operators have nonzero entropy.