Sean Carroll, seancarroll@gmail.com

125c, Lecture Five : 4/17/16

A note on operators ⇝ matrices.

Given an operator $\hat{\mathcal{O}} \in L(\mathcal{H})$ and a basis $\{|\phi_a\rangle\}$ for $\mathcal{H}$, matrix elements of $\hat{\mathcal{O}}$ are

$$\mathcal{O}_{ab} = \langle\phi_a|\hat{\mathcal{O}}|\phi_b\rangle .$$

Given the components of $\mathcal{O}_{ab}$ in some basis, we can instantly reconstruct the operator:

$$\hat{\mathcal{O}} = \sum_{ab} \mathcal{O}_{ab} |\phi_a\rangle\langle\phi_b| .$$

E.g. for a qubit in the $\{|0\rangle, |1\rangle\}$ basis, if $\rho_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then

$$\hat{\rho} = |0\rangle\langle 0| \qquad \text{etc.}$$

Note: just as a maximum-entropy distribution function is $f(x) = 1/N$, a maximum-entropy density operator is

$$\hat{\rho} = \frac{1}{d}\sum_a |\phi_a\rangle\langle\phi_a|, \quad \hat{\rho}_{ab} = \frac{1}{d}\mathbb{1}.$$

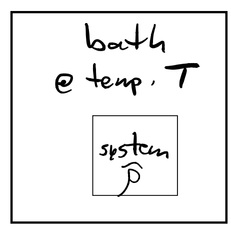
ρ_{ab} is diagonal, so

$$S = -\mathrm{Tr}\,\hat{\rho}\log\hat{\rho} = -\mathrm{Tr}\left[\left(\tfrac{1}{d}\mathbb{1}\right)\cdot\left(\log\tfrac{1}{d}\cdot\mathbb{1}\right)\right]$$

$$= \frac{1}{d}\cdot\log d\cdot\underbrace{\mathrm{Tr}\,\mathbb{1}}_{=d} = \log d.$$

$$\therefore\ d = \dim\mathcal{H} = e^{S_{max}}.$$

But often we have a thermal density operator, e.g. in contact with a heat bath (canonical ensemble, fixed $\langle H\rangle$):



$$\hat{\rho} = \frac{e^{-\hat{H}/kT}}{Z}, \qquad Z = \mathrm{Tr}\,e^{-\hat{H}/kT}$$

↳ "partition function"

Energy eigenbasis:

$$\hat{\rho} = \frac{1}{Z}\sum_a e^{-E_a/kT}|E_a\rangle\langle E_a|.$$

Maximum entropy if we fix $\langle \hat{H}\rangle_{\hat{\rho}}$.

Back to entropy of subsystems:

For $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, write in Schmidt decomposition: $|\Psi\rangle = \sum_n \sqrt{p_n}\, |\sigma_n\rangle_A |\lambda_n\rangle_B$

Then $\hat{\rho}_A = \sum_n p_n |\sigma_n\rangle_A\langle\sigma_n|$, $\hat{\rho}_B = \sum p_n |\lambda_n\rangle_B\langle\lambda_n|$.

$\therefore$ $S[\hat{\rho}_A] = S[\hat{\rho}_B]$, and $S[\hat{\rho}_{AB}] = 0$.

→ always true for pure states in $\mathcal{H}_A \otimes \mathcal{H}_B$.

S, the von Neumann entropy, is often called "entanglement entropy," but entanglement is not the only source.

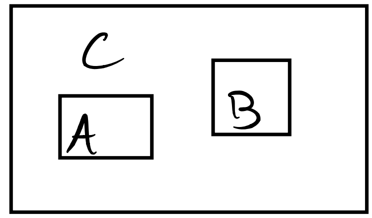
Indeed, $\hat{\rho}$ is a pure state iff it only has one nonzero eigenvector.

If $\hat{\rho}$ is not diagonal, it's hard to see if it's pure or mixed by inspection.

E.g. $\hat{\rho} = |+\rangle\langle+| = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$: pure, $S = 0$.

What about when the overall state $\hat{\rho}_{AB}$ is mixed, not pure?

We have the mutual information between A & B:



$$I(A:B) = S[\hat{\rho}_A] + S[\hat{\rho}_B] - S[\hat{\rho}_{AB}]$$

Mutual info = "What knowing A tells you about B (and vice-versa)."

If $I(A:B) = 0$, A & B might be entangled with the outside world, but they are not entangled with each other.

If $S[\hat{\rho}_{AB}] = 0$, then $S[\hat{\rho}_A] = S[\hat{\rho}_B]$,

and $I(A:B) = 2S[\hat{\rho}_A] = 2S[\hat{\rho}_B]$.

AB are unentangled with the outside world, though they may be entangled with each other.

Think of it as

$$\underbrace{S_A + S_B}_{\text{entropy of subsystems}} = \underbrace{S_{AB}}_{\text{total entropy}} + \underbrace{I(A:B)}_{\text{mutual information}}.$$

The physics of measurement.

[Preskill Ph 219 notes, Ch. 3]

Usual story: we have a state $|\Psi\rangle \in \mathcal{H}$, and some operator $\hat{X}$ we want to measure.

Find eigenstates $|x_n\rangle$: $\hat{X}|x_n\rangle = x_n|x_n\rangle$

Then ① we measure value x_n with probability $p(x_n) = |\langle x_n|\Psi\rangle|^2$, and

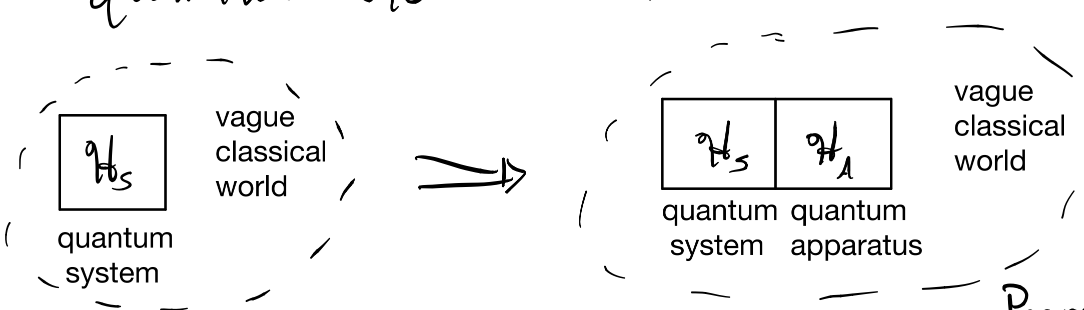
② afterward the state "collapses"

$$|\Psi\rangle \to |x_n\rangle .$$

A bit magical. Can we make it seem more like physics?

First (small) step by <u>von Neumann</u>.

Idea: treat the measuring apparatus (or "device" or "pointer") as a quantum system itself.



Progress!

(Dividing line between quantum & classical descriptions is the "Heisenberg Cut.")

System + apparatus $= \mathcal{H}_S \otimes \mathcal{H}_A$.

Bases: $\mathcal{H}_S$ – $\{|x_n\rangle_S\}$, $n \in \{1 \ldots d_S\}$

$\mathcal{H}_A$ – $\{|q_a\rangle_A\}$, $a \in \{1 \ldots d_A\}$

Start in an unentangled state. System is in a superposition of $|x_n\rangle$'s, but apparatus is in a "ready" state:

$$|\Psi(0)\rangle = |\Phi_S\rangle |\Omega_A\rangle$$
$$= \left(\sum_n \phi_n |x_n\rangle\right) |q_0\rangle .$$

No magic, so there is some Hamiltonian governing their evolution:

$$\hat{H} = \hat{H}_S + \hat{H}_A + \hat{H}_I$$

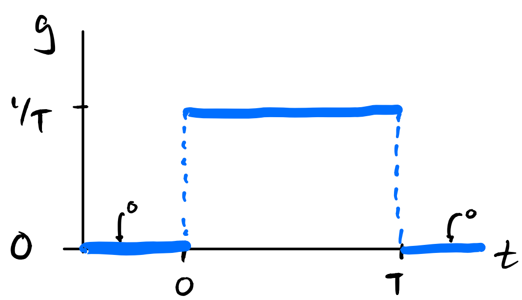
interaction.

It's the interaction that will matter.

Choose a simple form:

$$\hat{H}_I = g(t)\,\underset{\substack{\uparrow \\ \text{system quantity} \\ \text{we are measuring}}}{\hat{X}_S} \otimes \underset{\substack{\uparrow \\ \text{acts on} \\ \text{apparatus}}}{\hat{O}_A}$$

$g(t)$ is a "bump function" representing the fact that the interaction turns on & off:



$$\int_0^T g(t)\,dt = 1$$

Really we have, e.g., a particle moving through a Stern-Gerlach magnetic field, so $g(t)$ is just a convenient device.

We've represented the system state $|\Phi\rangle$ in the basis of eigenvectors of the operator we are observing:

$$\hat{X}_S |x_n\rangle = x_n |x_n\rangle.$$

$$\Longrightarrow \quad \hat{X}_S = \sum_n x_n |x_n\rangle\langle x_n| .$$

For further convenience, consider measurements that don't affect the system property being observed (other than to measure it).

A simple way to make that happen is if $\hat{X}_s$ commutes with the system's self-Hamiltonian:

$$[\hat{H}_s, \hat{X}_s] = 0.$$

Then we can basically ignore $\hat{H}_s$ & $\hat{H}_A$.

What about the apparatus? We want to shift the position of some "pointer."

If the pointer position is $\hat{Q}_A$, it gets shifted by its conjugate momentum $\hat{P}_A$, with

$$[\hat{Q}_A, \hat{P}_A] = i. \qquad (\hbar = 1.)$$

Then we have

$$e^{-ix\hat{P}_A}|q_a\rangle = |q_{a+x}\rangle. \qquad (\text{mod } d_A.)$$

So the right interaction Hamiltonian is

$$\hat{H} = \hat{H}_I = g(t)\,\hat{X}_s \otimes \hat{P}_A$$

← pointer momentum.

What does this Hamiltonian do to our state?

$$|\Psi(T)\rangle = \hat{U}(T)\,|\Psi(0)\rangle,$$

where

$$\begin{aligned}
\hat{U}(T) &= e^{-i\int \hat{H}_I\,dt} \\
&= e^{-i\int_0^T g(t)\,\hat{X}_S \hat{P}_A\,dt} \\
&= e^{-i\hat{X}_S \hat{P}_A} \\
&= e^{-i\left(\sum_n x_n |x_n\rangle\langle x_n|\right)\hat{P}_A} \\
&= \sum_n |x_n\rangle\langle x_n|\, e^{-ix_n \hat{P}_A}
\end{aligned}$$

To justify pulling $|x_n\rangle\langle x_n| = \hat{\Pi}_{x_n}$ down,

note $[\hat{\Pi}_{x_n}, \hat{P}_A] = 0$ and $(\hat{\Pi}_{x_n})^2 = \hat{\Pi}_{x_n}$.

$\left(e^x = 1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)$

So $\hat{U}(T)|\Psi(0)\rangle$

$$\begin{aligned}
&= \left[\sum_n |x_n\rangle\langle x_n| e^{-ix_n\hat{P}_A}\right]\left(\sum_m \phi_m |x_m\rangle\right)|q_0\rangle \\
&= \sum_{nm} |x_n\rangle\, \delta_{nm}\, \phi_m\, |q_{x_n}\rangle \\
&= \sum_n \phi_n |x_n\rangle |q_{x_n}\rangle.
\end{aligned}$$

In sum: $\left(\sum_n \phi_n |x_n\rangle\right)|q_0\rangle \underset{\text{evolve}}{\rightsquigarrow} \sum_n \phi_n |x_n\rangle |q_{x_n}\rangle$.

Under unitary evolution, an initially unentangled state with apparatus in "ready" position evolves to one where the state is entangled, matching eigenstates of the system observable with pointer eigenstates that have been shifted by the observed value x_n.

Von Neumann shows how an apparatus can "measure" a system by becoming entangled. But there is still magic — How do we measure the apparatus? Why is $p(x_n) = |\phi_n|^2$? Where is "collapse"?

The apparatus is supposed to be big and macroscopic, so maybe it's just obvious. Or maybe not.