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Generalized Measurements & POVM's

Von Neumann Measurement Formalism:

$$\hat{U}(T): |\Psi(0)\rangle = |\Phi_0\rangle_S |\eta_0\rangle_A = \left(\sum_n \phi_n |\phi_n\rangle_S\right)|\eta_0\rangle_A$$

$$\rightsquigarrow |\Psi(T)\rangle = \sum_n \phi_n |\phi_n\rangle_S |\eta_n\rangle_S$$

This standard measurement story has the property that the system states we measure, $\{|x_n\rangle\}$, are orthogonal to each other. The corresponding operators $\hat{\Pi}_{x_n} = |x_n\rangle\langle x_n|$ are projectors, $\hat{\Pi}_{x_n}\hat{\Pi}_{x_m} = \delta_{nm}\hat{\Pi}_{x_n}$.

We have a "Projection-Valued Measure", or PVM.

The upshot of a PVM is that the measurement unitary takes the form

$$\hat{U}(T) = \sum_{n,a} \hat{\Pi}_{x_n} \otimes |q_{a+x_n}\rangle\langle q_a|.$$

"In each component $|x_n\rangle$ of the system, shift the apparatus $a \to a + x_n$."

Doesn't have to be that way:

POVM's = Positive-Operator-Valued Measures.

These come from "generalized measurements."
Let's see how that goes.

The basis states $\{|q_a\rangle\}$ for our pointer served double duty:

1. They become entangled with corresponding states of the system
2. They are what we actually "measure" at the end of the day.

We can decouple these two roles.

Imagine we measure a spin, with our pointer in its standard basis:

$$\hat{U} : (\alpha|\uparrow\rangle_S + \beta|\downarrow\rangle_S) \otimes |0\rangle_A$$
$$\rightarrow |\Psi(T)\rangle = \alpha|\uparrow\rangle_S|0\rangle_A + \beta|\downarrow\rangle_S|1\rangle_A.$$

But let's read out the pointer in a different basis, e.g.

$$|\pm\rangle_A = \frac{1}{\sqrt{2}}\left(|0\rangle_A \pm |1\rangle_A\right).$$

By "read out the pointer," we mean project $|\Psi(T)\rangle$ onto $|\pm\rangle_A$. We have

$$\hat{\Pi}_+|0\rangle = |+\rangle\langle+|0\rangle = \frac{1}{2}\left(|0\rangle+|1\rangle\right)\left(\langle 0|+\langle 1|\right)|0\rangle$$

$$= \frac{1}{2}\left(|0\rangle+|1\rangle\right) = \frac{1}{\sqrt{2}}|+\rangle.$$

indeed, $$\hat{\Pi}_\pm \begin{Bmatrix}|0\rangle \\ |1\rangle\end{Bmatrix} = \begin{matrix}1/\sqrt{2}\,|\pm\rangle \\ \pm 1/\sqrt{2}\,|\pm\rangle.\end{matrix}$$

So projecting our time-evolved state gives

$$\hat{\Pi}_+^{(A)}|\Psi(T)\rangle = \alpha|\uparrow\rangle\left(\frac{1}{\sqrt{2}}|+\rangle\right) + \beta|\downarrow\rangle\left(\frac{1}{\sqrt{2}}|+\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(\alpha|\uparrow\rangle+\beta|\downarrow\rangle\right)|+\rangle$$

and $\hat{\Pi}_-^{(A)}|\Psi(T)\rangle = \frac{1}{\sqrt{2}}\left(\alpha|\uparrow\rangle-\beta|\downarrow\rangle\right)|-\rangle$.

So post-measurement, the system is in states

$\alpha|\uparrow\rangle \pm \beta|\downarrow\rangle$. Not orthogonal!
(unless $|\alpha| = |\beta|$.)

$$\left(\alpha^*\langle\uparrow|+\beta^*\langle\downarrow|\right)\left(\alpha|\uparrow\rangle-\beta|\downarrow\rangle\right) = |\alpha|^2-|\beta|^2.$$

Because not orthogonal, we can repeat a measurement right away and get a different result.

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Reading the apparatus in a basis that is <u>not</u> the post-interaction Schmidt basis is equivalent to doing non-orthogonal measurements of the system.

We can think directly in terms of <u>generalized measurements</u> on the system. Consider measurement unitaries of the form

$$\hat{U} = \sum_a \hat{M}_a \otimes |\eta_a\rangle\langle\eta_0|$$

not necessarily orthogonal.

← simplify by only considering action of $\hat{U}$ on the ready state $|\eta_0\rangle$.

$\{|\eta_a\rangle\} \in \mathcal{H}_A$ is an orthonormal basis:

$$\langle\eta_a|\eta_b\rangle = \delta_{ab}.$$

The measurement operators $\hat{M}_a$ act on the system, and the pointer is shifted $\eta_0 \to \eta_a$.

$$\hat{U} : |\Phi_0\rangle_S \otimes |\eta_0\rangle_A \to |\Psi(T)\rangle = \sum_a (\hat{M}_a |\Phi_0\rangle) |\eta_a\rangle$$

Pointer readouts $|\eta_a\rangle$ are entangled with system states $\hat{M}_a|\Phi_0\rangle$, but the $\{\hat{M}_a\}$ need not be orthogonal projectors.

Born Rule says the probability of the pointer reading $|\eta_a\rangle$ is

$$p(\eta_a) = \| \hat{\Pi}_{\eta_a} |\Psi(T)\rangle \|^2 = \| \hat{M}_a |\Phi_0\rangle |\eta_a\rangle \|^2$$
$$= | \langle \Phi_0 | \hat{M}_a^\dagger \hat{M}_a | \Phi_0 \rangle |^2 ,$$

and the collapse postulate says that the post-measurement state of the system is

$$\frac{\hat{M}_a |\Phi_0\rangle}{\| \hat{M}_a |\Phi_0\rangle \|} .$$

E.g. when we measured our pointer in the $|\eta_a\rangle = |\pm\rangle$ basis, the evolution was

$$\hat{U} : (\alpha|\uparrow\rangle + \beta|\downarrow\rangle)|0\rangle \to \sum_a \left[\hat{M}_a (\alpha|\uparrow\rangle + \beta|\downarrow\rangle) \right] |\eta_a\rangle$$
$$= \tfrac{1}{\sqrt{2}} (\alpha|\uparrow\rangle + \beta|\downarrow\rangle)|+\rangle + \tfrac{1}{\sqrt{2}} (\alpha|\uparrow\rangle - \beta|\downarrow\rangle)|-\rangle .$$

So in this example, the measurement matrices $\hat{M}_a$ are

$$\hat{M}_+ = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = \tfrac{1}{\sqrt{2}} \mathbb{1} , \quad \hat{M}_+ = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \tfrac{1}{\sqrt{2}} Z .$$

A POVM with $\hat{U} = \sum_a \hat{M}_a \otimes |\eta_a\rangle\langle\eta_0|$ leaves the observed system in the state $\hat{M}_a |\Phi_0\rangle$ (up to normalization) when the pointer reads $|\eta_a\rangle$.

We know that $\hat{U}$ is unitary, which implies a <u>completeness relation</u> for the $\hat{M}_a$. (6)

To see this, note $\| \hat{U} |\Psi(0)\rangle \|^2 = 1$, so

$$\hat{U}|\Psi(0)\rangle = \left(\sum_a \hat{M}_a \otimes |\eta_a\rangle\langle\eta_0| \right) |\Phi_0\rangle|\eta_0\rangle$$

$$= \left(\sum_a \hat{M}_a |\Phi_0\rangle \right) |\eta_a\rangle$$

Norm:

$$\| \hat{U}|\Psi(0)\rangle \|^2 = \langle \Psi(0)| \hat{U}^\dagger \hat{U} |\Psi(0)\rangle$$

$$= \left(\sum_a \langle\Phi_0|\langle\eta_a| \hat{M}_a^\dagger \right) \left(\sum_b \hat{M}_b |\Phi_0\rangle|\eta_b\rangle \right)$$

$$= \sum_{ab} \langle \Phi_0 | \hat{M}_a^\dagger \hat{M}_b |\Phi_0\rangle \underbrace{\langle \eta_a | \eta_b \rangle}_{\delta_{ab}}$$

$$= \sum_a \langle \Phi_0 | \hat{M}_a^\dagger \hat{M}_a |\Phi_0\rangle .$$

This holds for any $|\Phi_0\rangle$, meaning

$$\sum_a \hat{M}_a^\dagger \hat{M}_a = \mathbb{1} .$$ → Completeness Relation.

This assures that probabilities add to 1:

$$\sum_a p(\eta_a) = \sum_a |\langle \Phi_0 | \hat{M}_a^\dagger \hat{M}_a | \Phi_0 \rangle|^2$$
$$= |\langle \Phi_0 | \left(\sum_a \hat{M}_a^\dagger M_a \right) | \Phi_0 \rangle|^2$$
$$= |\langle \Phi_0 | \Phi_0 \rangle| = 1 .$$

It also suggests that we simply define

$$\hat{E}_a = \hat{M}_a^\dagger \hat{M}_a ,$$

so that $\quad \sum_a \hat{E}_a = \mathbb{1} .$

The set of operators $\{\hat{E}_a\}$, acting on $\mathcal{H}_S$, is what is often called "a POVM."

Sometimes it is most convenient to use the measurement operators $\hat{M}_a$.

They are complex square matrices obeying the completeness relation $\sum_a \hat{M}_a^\dagger \hat{M}_a$, otherwise unconstrained.

The $\hat{M}_a$'s are examples of "Kraus operators," to be discussed next lecture. Useful in describing the dynamics of open quantum systems.

Other times it's useful to use the POVM matrices $\hat{E}_a = \hat{M}_a^\dagger \hat{M}_a$.

Properties:

- Hermitian. $\hat{E}_a^\dagger = \hat{E}_a$.
- Positive. $\langle\psi|\hat{E}_a|\psi\rangle \geq 0 \quad \forall\, |\psi\rangle$. ("$\hat{E}_a > 0$.")
- Complete. $\sum_a \hat{E}_a = \mathbb{1}$.

Like a set of projectors, but $\hat{E}_a \hat{E}_b \neq \delta_{ab} \hat{E}_a$ (in general).

Note that, given $\hat{E}_a$'s, $\hat{M}_a$'s aren't unique. Invariant under unitary transformations $\hat{M}_a \to \hat{U}\hat{M}_a$:

$$\hat{E}_a = \hat{M}_a^\dagger \hat{M}_a = \hat{M}_a^\dagger \underbrace{\hat{U}^\dagger \hat{U}}_{=\mathbb{1}} \hat{M}_a$$

<u>Upshot</u>. (Check for yourself!)

Given a system with density operator $\hat{\rho}_s$ and a POVM $\{\hat{E}_a\}$, interaction with the measuring apparatus produces evolution

$$\hat{\rho}_s(0) \longrightarrow \hat{\rho}_s(T) = \sum_a \hat{E}_a \hat{\rho}_s(0) \hat{E}_a .$$

The probability of observing measurement outcome "a" is

$$p(a) = \mathrm{Tr}\left(\hat{\rho}_s \hat{E}_a\right).$$

After the measurement is performed and the pointer is read out (what some might call "wave function collapse"), the new system density operator is

$$\hat{\rho}_s(T, a) = \frac{\hat{M}_a^{+} \hat{\rho}_s(0) \hat{M}_a}{\mathrm{Tr}\left[\hat{M}_a^{+} \hat{\rho}_s(0) \hat{M}_a\right]} , \quad \hat{M}_a^{+} \hat{M}_a = \hat{E}_a .$$

Measuring again, even right away, does <u>not</u> necessarily return the same result.

Let's do an example before we lose our minds.

Alice sends some qubits to Bob.

But her qubit machine is broken, and can only send either $|0\rangle$ or $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

Obviously these are non-orthogonal:

$$\langle 0 | + \rangle = \frac{1}{\sqrt{2}}.$$

Can Bob read the message?

If he simply measures in the $|0\rangle/|1\rangle$ basis, when he obtains $|1\rangle$ he knows for sure the qubit was $|+\rangle$, but if he obtains $|0\rangle$ he can't be sure ($|+\rangle$ would yield that outcome 50% of the time).

Maybe the message is really important.

Is there a strategy where

1. Bob has a nonzero chance of identifying <u>either</u> qubit, and
2. He never <u>misidentifies</u> a qubit?

(Allowed to judge that a certain qubit was just unreadable.)

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<u>Yes</u>: construct POVM measurement operators from states <u>orthogonal</u> to $|0\rangle$, $|+\rangle$.

$$\hat{E}_1 = k\, |1\rangle\langle 1|$$

$$\hat{E}_2 = k\, |-\rangle\langle -| = \frac{k}{2}\left(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|\right)$$

$$\hat{E}_3 = \mathbb{1} - \hat{E}_1 - \hat{E}_2$$

$$= \mathbb{1} - \frac{k}{2}\left(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| - |1\rangle\langle 1|\right).$$

To keep $\hat{E}_3 > 0$, require $k = 1/(1 + 1/\sqrt{2}) = 0.59$ (or less).

then the readout probabilities are

$p(E_i, |\psi\rangle) = \langle\psi|\hat{E}_i|\psi\rangle$, or

	E_1	E_2	E_3
$\|0\rangle$	0	$k/2 = 0.29$	$1 - k/2 = 0.71$
$\|+\rangle$	$\frac{k}{2} = 0.29$	0	$1 - k/2 = 0.71$

If Bob's pointer says E_1, he is certain the qubit was $|+\rangle$; likewise $E_2 \to |0\rangle$.

E_3 happens 71%, and then Bob doesn't know. But he knows he doesn't know! So he throws away that measurement, and never makes a mistake.