Sean Carroll, seancarroll@gmail.com

125c, Lecture Seven : 4/24/17

Open Systems & Quantum Channels

[Schumacher and Nielsen-Chuang are good references here, in addition to Preskill.]

We often have a composite system, say $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, and we want to keep track of subsystem A without worrying (too much) about B.

It makes sense to study the reduced density matrix,

$$\hat{\rho}_A = \mathrm{Tr}_B\, \hat{\rho}\,.$$

A closed system obeys the von Neumann eq., $\partial_t \hat{\rho} = -i[\hat{H}, \hat{\rho}]$, but generally $\hat{\rho}_A$ describes an open system (it interacts with B) and we have to think harder.

Two plausible lines of attack:

1. Start with evolution of closed system $\hat{\rho} \in L(\mathcal{H}_A \otimes \mathcal{H}_B)$, then reduce to evolution of $\hat{\rho}_A \in L(\mathcal{H}_A)$.
2. Construct general rules that evolution of $\hat{\rho}_A$ must obey (always, or under reasonable assumptions).

We'll jump back & forth between these strategies.

Time evolution (for some fixed interval T) takes one density operator to another one. Think of this as a "superoperator":

$$\mathcal{E} : \hat{\rho}(t=0) \longrightarrow \hat{\rho}(t=T).$$

↑ no subscripts b/c we're being general.

Also called a "quantum channel", after the idea of sending/transforming some quantum information.

Properties a quantum channel must have:

1. Linearity: $\mathcal{E}(a\hat{\rho}_1 + b\hat{\rho}_2) = a\,\mathcal{E}(\hat{\rho}_1) + b\,\mathcal{E}(\hat{\rho}_2)$.
2. Preserves Hermiticity: $\hat{\rho}^\dagger = \hat{\rho} \rightarrow \mathcal{E}(\hat{\rho})^\dagger = \mathcal{E}(\hat{\rho})$.
3. Trace-Preserving: $\mathrm{Tr}[\mathcal{E}(\hat{\rho})] = \mathrm{Tr}\{\hat{\rho}]$.
4. Preserves Positivity: $\hat{\rho} \geq 0 \rightarrow \mathcal{E}(\hat{\rho}) \geq 0$.

Because of these, quantum channels are also sometimes called "Completely Positive Trace-Preserving (CPTP) Maps."

["Completely" positive is a technicality, because we want positivity even if $\hat{\rho}$ is a subsystem of a larger system. I.e, we demand that $\hat{\rho}_A \otimes \mathbb{1}_B \geq 0 \rightarrow \mathcal{E}(\hat{\rho}_A) \otimes \mathbb{1}_B \geq 0$.]

Final jargon point: a superoperator that does not preserve the trace (but does preserve positivity & Hermiticity) is sometimes called a "quantum operation" (as opposed to channel). But usage isn't uniform, so be careful.

An obvious example of a quantum channel is unitary evolution of a closed system: ④

$$\mathcal{E}_u(\hat{\rho}) = \hat{U}\hat{\rho}\hat{U}^\dagger.$$

A less trivial example is a POVM on a subsystem.

Recall $\hat{U}(T): |\Psi(0)\rangle = |\Phi_0\rangle_S \otimes |\eta_0\rangle_A$

$$\rightsquigarrow |\Psi(T)\rangle = \sum_a \hat{M}_a |\Phi_0\rangle_S \otimes |\eta_a\rangle.$$

Reduced density matrix for system:

$$\hat{\rho}_S(0) = \mathrm{Tr}_A |\Psi(0)\rangle\langle\Psi(0)| = |\Phi_0\rangle_S\langle\Phi_0|$$

$$\begin{aligned}
\hat{\rho}_S(T) &= \mathrm{Tr}_A |\Psi(T)\rangle\langle\Psi(T)| \\
&= \sum_a \langle\eta_a| \left(\sum_b \hat{M}_b |\Phi_0\rangle|\eta_b\rangle\right)\left(\sum_c \langle\eta_c|\langle\Phi_0|\hat{M}_c^\dagger\right)|\eta_a\rangle \\
&= \sum_a \hat{M}_a |\Phi_0\rangle\langle\Phi_0| \hat{M}_a^\dagger \\
&= \sum_a \hat{M}_a \hat{\rho}_S(0) \hat{M}_a^\dagger .
\end{aligned}$$

Thus a POVM defines a quantum channel:

$$\mathcal{E}_{POVM}(\hat{\rho}_s) = \sum_a \hat{M}_a \hat{\rho}_s \hat{M}_a^\dagger . \quad (*)$$

That is, of course, the system density operator before we observe the pointer. Afterward we "collapse" onto pointer readout $|\eta_a\rangle$, and we can think of POVM + collapse as yet another quantum channel on the system:

$$\mathcal{E}_{M_a}(\hat{\rho}_s) = \frac{\hat{M}_a \hat{\rho}_s \hat{M}_a^\dagger}{\mathrm{Tr}[\hat{M}_a^\dagger \hat{\rho}_s \hat{M}_a]}$$

This occurs with probability

$$p(a) = \mathrm{Tr}[\hat{M}_a \hat{\rho}_s \hat{M}_a^\dagger] .$$

The expression ⊛ for the POVM superoperator is one example of something more general: the operator-sum representation.

Consider the situation of $\hat{\rho} \in L(\mathcal{H}_A \otimes \mathcal{H}_B)$, where we start with A & B initially unentangled and in pure states $|\alpha\rangle_A$ and $|\beta\rangle_B$, then evolve unitarily.

$$\hat{\rho}(0) = |\alpha\rangle_A\langle\alpha| \otimes |\beta\rangle_B\langle\beta|$$

$$\hat{\rho}(T) = \hat{U}(T)\left[|\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta|\right]\hat{U}^{+}(T)$$

$$\hat{\rho}_A(T) = \mathrm{Tr}_B\left[\hat{U}(T)\left(|\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta|\right)\hat{U}^{+}(T)\right].$$

Think of this as a quantum channel:

$$\mathcal{E}(\hat{\rho}_A(0)) = \hat{\rho}_A(T).$$

Clearly the channel will, in general, depend on the environment state $|\beta\rangle$ as well as the unitary $\hat{U}(T)$.

Now choose a basis $\{|\eta_b\rangle_B\} \in \mathcal{H}_B$,

so that $\mathrm{Tr}_B \hat{\rho} = \sum {}_B\langle\eta_b|\hat{\rho}|\eta_b\rangle_B$.

Define Kraus operators $\hat{K}_b \in L(\mathcal{H}_A)$ by

$$\hat{K}_b \equiv {}_B\langle\eta_b|\,\hat{U}(T)\,|\beta\rangle_B .$$

(an operator on $\mathcal{H}_A$, not a number.)

(assumed initial pure state of B).

Then

$$\mathcal{E}(\hat{\rho}_A(0)) = \sum_b {}_B\langle\eta_b|\,\hat{U}(T)\left[|\alpha\rangle_A\langle\alpha| \otimes |\beta\rangle_B\langle\beta|\right]\hat{U}^\dagger(T)|\eta_b\rangle_B$$

$$= \sum_b \hat{K}_b\,|\alpha\rangle_A\langle\alpha|\,\hat{K}_b^\dagger .$$

(POVM's are an obvious example.)

Note that

① Choosing $\hat{\rho}_B$ to be pure wasn't really a restriction; if $\hat{\rho}_B$ were not pure, we could imagine expanding Hilbert space and purifying it.

② Choosing $\hat{\rho}_A$ to be pure wasn't really a restriction either, since any $\hat{\rho}_A$ is $\sum_a p_a|\alpha_a\rangle\langle\alpha_a|$ for some $|\alpha_a\rangle$'s, and $\mathcal{E}$ is linear.

<u>An important result</u>: when dynamics for $\hat{\rho}_A \in L(\mathcal{H}_A)$ result from unitary dynamics for $\hat{\rho} \in L(\mathcal{H}_A \otimes \mathcal{H}_B)$, we can always write the corresponding quantum channel as

$$\mathcal{E}(\hat{\rho}_A) = \sum_b \hat{K}_b \hat{\rho}_A \hat{K}_b^\dagger .$$

This is the <u>Kraus representation</u> or <u>operator-sum representation</u> of the quantum channel, and the $\hat{K}_a$'s are Kraus operators. The CPTP conditions imply a completeness relation,

$$\sum_b \hat{K}_b^\dagger \hat{K}_b = \mathbb{1} .$$

Converse is also true: a superoperator written in terms of Kraus operators will satisfy CPTP conditions.

Note that, as we saw with POVM measurement operators $\hat{M}_a$, Kraus operators are not unique. We derived them after choosing a basis $\{|\eta_b\rangle\} \in \mathcal{H}_B$, so it shouldn't be surprising that we can do an arbitrary unitary transformation

$$\hat{K}_b \to \hat{U}\hat{K}_b .$$

Even the number of Kraus operators for a given channel isn't uniquely defined.

There is a nice physical interpretation of the operator-sum representation.

$$\mathcal{E}(\hat{\rho}) = \sum_b \hat{K}_b \hat{\rho} \hat{K}_b^+ .$$

Given our earlier discussion of POVM's, this says "Take the density operator $\hat{\rho}$, and replace it with a sum of operators $(\hat{K}_b\hat{\rho}\hat{K}_b^+)/\mathrm{Tr}(\hat{K}_b\hat{\rho}\hat{K}_b^+)$, with weight (probability) $\mathrm{Tr}(\hat{K}_b\hat{\rho}\hat{K}_b^+)$." As if the system had been measured, but the outcome unknown.

Example: Dephasing Channel.

Consider a qubit A in the $|0\rangle, |1\rangle$ basis, and a large "environment" E.

Imagine the environment starts in the state $|0\rangle_E$, and we have the following unitary evolution:

$$\hat{U}: \begin{cases} |0\rangle_A |0\rangle_E \to \sqrt{1-p}\, |0\rangle_A |0\rangle_E + \sqrt{p}\, |0\rangle_A |1\rangle_E \\ |1\rangle_A |0\rangle_E \to \sqrt{1-p}\, |1\rangle_A |0\rangle_E + \sqrt{p}\, |1\rangle_A |2\rangle_E . \end{cases}$$

Note that our qubit itself "doesn't evolve" — only its entanglement does. (That's a fake distinction, since there is only one wave function.) With probability $(1-p)$, nothing changes; with probability p, qubit states $|0\rangle$ and $|1\rangle$ become entangled with environment states $|1\rangle$ and $|2\rangle$, respectively. (Maybe a photon in the environment scattered off our qubit.)

There will be three Kraus operators, corresponding to the three environment states $\{|\eta_b\rangle_E\} = \{|0\rangle_E, |1\rangle_E, |2\rangle_E\}$.

Write the unitary time-evolution operator as

$$\hat{U}(T) = \sqrt{1-p}\,|0\rangle_A|0\rangle_E\langle 0|_E\langle 0|_A + \sqrt{p}\,|0\rangle_A|1\rangle_E\langle 0|_E\langle 0|_A$$
$$+\sqrt{1-p}\,|1\rangle_A|0\rangle_E\langle 0|_E\langle 1|_A + \sqrt{p}\,|1\rangle_A|2\rangle_E\langle 0|_E\langle 1|_A + \cdots$$

(Incomplete, but all we need to act on $|0\rangle|0\rangle$ and $|0\rangle|1\rangle$.)

The Kraus operators are therefore

$$\hat{K}_b = {}_E\langle\eta_b|\,\hat{U}\,|0\rangle_E$$

$$\hat{K}_0 = \sqrt{1-p}\,|0\rangle_A\langle 0| + \sqrt{1-p}\,|1\rangle_A\langle 1| = \sqrt{1-p}\,\mathbb{1}$$

$$\hat{K}_1 = \sqrt{p}\,|0\rangle_A\langle 0| = \sqrt{p}\,\hat{\Pi}_0 = \sqrt{p}\begin{pmatrix}1&0\\0&0\end{pmatrix}$$

$$\hat{K}_2 = \sqrt{p}\,|1\rangle_A\langle 1| = \sqrt{p}\,\hat{\Pi}_1 = \sqrt{p}\begin{pmatrix}0&0\\0&1\end{pmatrix}.$$

You can verify: $\sum_b \hat{K}_b^\dagger \hat{K}_b = \mathbb{1}$.

Quantum channel:

$$\mathcal{E}(\hat{\rho}) = \sum_b \hat{K}_b \hat{\rho} \hat{K}_b^+$$

$$= (1-p)\hat{\rho} + p\begin{pmatrix} \rho_{00} & 0 \\ 0 & 0 \end{pmatrix} + p\begin{pmatrix} 0 & 0 \\ 0 & \rho_{11} \end{pmatrix}$$

$$= \begin{pmatrix} \rho_{00} & (1-p)\rho_{01} \\ (1-p)\rho_{10} & \rho_{11} \end{pmatrix}.$$

p is a real number between 0 and 1, so diagonal terms remain constant, while the off-diagonal terms decay (in this basis).

Imagine waiting N time steps:

$$\mathcal{E}^N(\hat{\rho}) = \begin{pmatrix} \rho_{00} & (1-p)^N\rho_{01} \\ (1-p)^N\rho_{10} & \rho_{11} \end{pmatrix} \longrightarrow \begin{pmatrix} \rho_{00} & 0 \\ 0 & \rho_{11} \end{pmatrix}$$

Loss of phase information diagonalizes the density matrix (in this basis).

A simple example of decoherence.