We often have a composite system, say \( \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \), and we want to keep track of subsystem \( A \) without worrying (too much) about \( B \).

It makes sense to study the reduced density matrix

\[
\hat{\rho}_A = \text{Tr}_B \hat{\rho}.
\]

A closed system obeys the von Neumann eq., \( \dot{\hat{\rho}} = -i [\hat{H}, \hat{\rho}] \), but generally \( \hat{\rho}_A \) describes an open system (it interacts with \( B \)) and we have to think harder.
Two plausible lines of attack:

1. Start with evolution of closed system $\hat{\rho} \in L(\mathcal{H}_A \otimes \mathcal{H}_B)$, then reduce to evolution of $\hat{\rho}_A \in L(\mathcal{H}_A)$.

2. Construct general rules that evolution of $\hat{\rho}_A$ must obey (always, or under reasonable assumptions).

We'll jump back & forth between these strategies.

Time evolution (for some fixed interval $T$) takes one density operator to another one. Think of this as a "superoperator":

$$\mathcal{E} : \hat{\rho}(t=0) \rightarrow \hat{\rho}(t=T).$$

(No subscripts b/c we're being general.)

Also called a "quantum channel", after the idea of sending/transforming some quantum information.
Properties a quantum channel must have:

1. Linearity: \( \mathcal{E}(a \hat{\rho}_1 + b \hat{\rho}_2) = a \mathcal{E}(\hat{\rho}_1) + b \mathcal{E}(\hat{\rho}_2) \).
2. Preserves Hermiticity: \( \hat{\rho}^\dagger = \hat{\rho} \rightarrow \mathcal{E}(\hat{\rho})^\dagger = \mathcal{E}(\hat{\rho}) \).
3. Trace-Preserving: \( \text{Tr}[\mathcal{E}(\hat{\rho})] = \text{Tr}[\hat{\rho}] \).
4. Preserves Positivity: \( \hat{\rho} \geq 0 \rightarrow \mathcal{E}(\hat{\rho}) \geq 0 \).

Because of these, quantum channels are also sometimes called "Completely Positive Trace-Preserving (CPTP) Maps."

["Completely" positive is a technicality, because we want positivity even if \( \hat{\rho} \) is a subsystem of a larger system. I.e., we demand that \( \hat{\rho}_A \otimes 1_B \geq 0 \rightarrow \mathcal{E}(\hat{\rho}_A) \otimes 1_B \geq 0 \).]

Final jargon point: a superoperator that does not preserve the trace (but does preserve positivity & Hermiticity) is sometimes called a "quantum operation" (as opposed to channel). But usage isn't uniform, so be careful.
An obvious example of a quantum channel is unitary evolution of a closed system:

$$\hat{E}_\sigma (\hat{\rho}) = \hat{U} \hat{\rho} \hat{U}^+.$$

A less trivial example is a POVM on a subsystem.

Recall $$\hat{U}(t) : |\Psi (0)\rangle = |\Psi_0\rangle \otimes |\eta_0\rangle_A \rightarrow |\Psi (t)\rangle = \sum_a \hat{M}_a |\Psi_0\rangle \otimes |\eta_0\rangle.$$ Reduced density matrix for system:

$$\hat{\rho}_S (0) = Tr_A |\Psi (0)\rangle \otimes |\Psi (0)\rangle = |\Psi_0\rangle \otimes |\Psi_0\rangle$$

$$\hat{\rho}_S (t) = Tr_A |\Psi (t)\rangle \otimes |\Psi (t)\rangle =$$

$$= \sum_a \langle \eta_a | (\sum_b \hat{M}_b |\eta_0\rangle \langle \eta_0 | \hat{M}_b^+ |\eta_0\rangle) \langle \eta_c | (\sum_c \langle \eta_0 | \hat{M}_c^+ |\eta_0\rangle |\eta_c\rangle$$

$$= \sum_a \hat{M}_a |\Psi_0\rangle \otimes |\Psi_0\rangle |\hat{M}_a^+$$

$$= \sum_a \hat{M}_a \hat{\rho}_S (0) \hat{M}_a^+.$$
Thus a POVM defines a quantum channel:

$$\mathcal{E}_{\text{POVM}}(\hat{\rho}_s) = \sum_a \hat{M}_a \hat{\rho}_s \hat{M}_a^+.$$  

That is, of course, the system density operator before we observe the pointer. Afterward we “collapse” onto pointer readout $|\eta_a\rangle$, and we can think of POVM collapse as yet another quantum channel on the system:

$$\mathcal{E}_{\eta_a}(\hat{\rho}_s) = \frac{\hat{M}_a \hat{\rho}_s \hat{M}_a^+}{\text{Tr}[\hat{M}_a^+ \hat{\rho}_s \hat{M}_a]}.$$  

This occurs with probability

$$p(a) = \text{Tr}[\hat{M}_a \hat{\rho}_s \hat{M}_a^+]$$.
The expression $\Theta$ for the POVM superoperator is one example of something more general: the operator-sum representation.

Consider the situation of $\hat{\sigma} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$, where we start with $A \& B$ initially unentangled and in pure states $|x\rangle_A$ and $|\beta\rangle_B$, then evolve unitarily.

$$\hat{\sigma}(\omega) = |x\rangle_A \otimes |\beta\rangle_B$$

$$\hat{\sigma}(\tau) = \hat{U}(\tau)[|x\rangle_A \otimes |\beta\rangle_B] \hat{U}^+(\tau)$$

$$\hat{P}_A(\tau) = \text{Tr}_B[\hat{U}(\tau) (|x\rangle_A \otimes |\beta\rangle_B) \hat{U}^+(\tau)].$$

Think of this as a quantum channel:

$$\mathcal{E}(\hat{P}_A(\omega)) = \hat{P}_A(\tau).$$

Clearly the channel will, in general, depend on the environment state $|\beta\rangle$ as well as the unitary $\hat{U}(\tau)$. 
Now choose a basis \( \{ \eta_b \}_b \in \mathcal{H}_B \), so that \( \text{Tr}_B \hat{\rho} = \sum_b \langle \eta_b | \hat{\rho} | \eta_b \rangle_B \).

Define Kraus operators \( \hat{K}_b \in \mathcal{L}(\mathcal{H}_A) \) by

\[
\hat{K}_b = \langle \eta_b | \hat{U}(T) | \beta \rangle_B.
\]

Then (assumed initial pure state of \( B \)).

\[
\epsilon(\hat{\rho}_A(0)) = \sum_b \langle \eta_b | \hat{U}(T) [1 \otimes \chi_b] \otimes \beta_X | \eta_b \rangle_B
\]

\[
= \sum_b \hat{K}_b [1 \otimes \chi_b] \hat{K}^+_b.
\]

(POVM's are an obvious example.)

Note that

1. Choosing \( \hat{\rho}_B \) to be pure wasn’t really a restriction; if \( \hat{\rho}_B \) were not pure, we could imagine expanding Hilbert space and purifying it.

2. Choosing \( \hat{\rho}_A \) to be pure wasn’t really a restriction either, since any \( \hat{\rho}_A \) is \( \sum_a \rho_a X_{a a} \) for some \( |a\rangle \)'s, and \( \epsilon \) is linear.
An important result: when dynamics for $\hat{\rho}_A \in \mathcal{L}(\mathcal{H}_A)$ result from unitary dynamics for $\hat{\sigma} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$, we can always write the corresponding quantum channel as

$$\mathcal{E}(\hat{\rho}_A) = \sum_b \hat{K}_b \hat{\rho}_A \hat{K}_b^+.$$ 

This is the Kraus representation or operator-sum representation of the quantum channel, and the $\hat{K}_b$'s are Kraus operators. The CPTP conditions imply a completeness relation:

$$\sum_b \hat{K}_b^+ \hat{K}_b = 1.$$ 

Conversely, it is also true: a superoperator written in terms of Kraus operators will satisfy CPTP conditions.
Note that, as we saw with POVM measurement operators $\hat{M}_a$, Kraus operators are not unique. We derived them after choosing a basis $\{|m_b\rangle \rangle \in \mathcal{H}_B$, so it shouldn’t be surprising that we can do an arbitrary unitary transformation $\hat{K}_b \rightarrow \hat{U}\hat{K}_b$. Even the number of Kraus operators for a given channel isn’t uniquely defined.

There is a nice physical interpretation of the operator-sum representation.

$$\mathcal{E}(\hat{\rho}) = \sum_b \hat{K}_b \hat{\rho} \hat{K}_b^\dagger.$$  

Given our earlier discussion of POVMs, this says “Take the density operator $\hat{\rho}$, and replace it with a sum of operators $(\hat{K}_b ^\dagger \hat{\rho} \hat{K}_b^\dagger)/\text{Tr}(\hat{K}_b \hat{\rho} \hat{K}_b^\dagger)$, with weight (probability) $\text{Tr}(\hat{K}_b \hat{\rho} \hat{K}_b^\dagger)$.” As if the system had been measured, but the outcome unknown.
Example: **Dephasing Channel.**

Consider a qubit $A$ in the $|0\rangle, |1\rangle$ basis, and a large “environment” $E$. Imagine the environment starts in the state $|0\rangle_E$, and we have the following unitary evolution:

$$\hat{U}: \begin{cases} 
|0\rangle_A |0\rangle_E \rightarrow \sqrt{1-p} |0\rangle_A |0\rangle_E + \sqrt{p} |0\rangle_A |1\rangle_E \\
|1\rangle_A |0\rangle_E \rightarrow \sqrt{1-p} |1\rangle_A |0\rangle_E + \sqrt{p} |1\rangle_A |1\rangle_E 
\end{cases}$$

Note that our qubit itself “doesn’t evolve”—only its entanglement does. (That’s a fake distinction, since there is only one wave function.) With probability $(1-p)$, nothing changes; with probability $p$, qubit states $|0\rangle$ and $|1\rangle$ become entangled with environment states $|0\rangle$ and $|1\rangle$, respectively. (Maybe a photon in the environment scattered off our qubit.)
There will be three Kraus operators, corresponding to the three environment states \( \{ \eta_b \}_{e^3} = \{ 10 \}_{e^3}, \{ 11 \}_{e^3}, \{ 12 \}_{e^3} \).

Write the unitary time-evolution operator as

\[
\hat{U}(T) = \sqrt{1 - p} \begin{pmatrix} |10\rangle_A \otimes |01\rangle_A \langle 01 | \\
|11\rangle_A \otimes |01\rangle_A \langle 01 | & |11\rangle_A \otimes |11\rangle_A \langle 11 | \end{pmatrix} + \sqrt{p} \begin{pmatrix} |10\rangle_A \otimes |11\rangle_A \langle 11 | & |11\rangle_A \otimes |12\rangle_A \langle 11 | \\
|11\rangle_A \otimes |11\rangle_A \langle 11 | & \sqrt{p} |12\rangle_A \otimes |12\rangle_A \langle 12 | \end{pmatrix} + \ldots
\]

(Incomplete, but all we need to act on \(10\rangle_{10}\) and \(10\rangle_{11}\).)

The Kraus operators are therefore

\[
\hat{K}_b = \langle \eta_b | \hat{U} | 10 \rangle_{e^3}
\]

\[
\hat{K}_0 = \sqrt{1 - p} |10\rangle_A \otimes |01\rangle_A + \sqrt{1 - p} |11\rangle_A \otimes |11\rangle_A = \sqrt{1 - p} \mathbb{1}
\]

\[
\hat{K}_1 = \sqrt{p} |10\rangle_A \otimes |11\rangle_A = \sqrt{p} \begin{pmatrix} 1 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \end{pmatrix} = \sqrt{p} \begin{pmatrix} 1 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \end{pmatrix}
\]

\[
\hat{K}_2 = \sqrt{p} |11\rangle_A \otimes |11\rangle_A = \sqrt{p} \begin{pmatrix} 0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \end{pmatrix} = \sqrt{p} \begin{pmatrix} 0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \end{pmatrix}
\]

You can verify: \( \sum_b \hat{K}_b^\dagger \hat{K}_b = \mathbb{1} \).
Quantum channel:

\[ \mathcal{E}(\rho) = \sum_b \hat{R}_b \hat{\rho} \hat{R}_b^+ \]

\[ = (1-p)\hat{\rho} + p \begin{pmatrix} \rho_{00} & 0 \\ 0 & 0 \end{pmatrix} + p \begin{pmatrix} 0 & 0 \\ 0 & \rho_{11} \end{pmatrix} \]

\[ = \begin{pmatrix} \rho_{00} & (1-p)\rho_{01} \\ (1-p)\rho_{10} & \rho_{11} \end{pmatrix}. \]

\( p \) is a real number between 0 and 1, so diagonal terms remain constant, while the off-diagonal terms decay (in this basis).

Imagine waiting \( N \) time steps:

\[ \mathcal{E}^N(\hat{\rho}) = \begin{pmatrix} \rho_{00} & (1-p)^N\rho_{01} \\ (1-p)^N\rho_{10} & \rho_{11} \end{pmatrix} \rightarrow \begin{pmatrix} \rho_{00} & 0 \\ 0 & \rho_{11} \end{pmatrix} \]

Loss of phase information diagonalizes the density matrix (in this basis).

A simple example of decoherence.