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125c, Lecture Eight: 4/26/17

Remember: Dephasing Channel.

Operator-sum expansion for Q. channel:

$$\mathcal{E}(\hat{\rho}) = \sum_b \hat{K}_b \hat{\rho} \hat{K}_b^\dagger$$

Kraus operators: $\hat{K}_b = {}_E\langle \eta_b | \hat{U} | 0 \rangle_E$

Dephasing: $\{|0\rangle_A, |1\rangle_A\} \in \mathcal{H}_A$, $\{|0\rangle_E, |1\rangle_E, |2\rangle_E \cdots\} \in \mathcal{H}_E$

With probability p an environment photon scatters off our qubit, $|00\rangle \to |01\rangle$ and $|10\rangle \to |12\rangle$.

$$\hat{U}(T) = \sqrt{1-p}\, |0\rangle_A |0\rangle_E \langle 0|_A\langle 0| + \sqrt{p}\, |0\rangle_A |1\rangle_E \langle 0|_A\langle 0|$$
$$+ \sqrt{1-p}\, |1\rangle_A |0\rangle_E \langle 0|_A\langle 1| + \sqrt{p}\, |1\rangle_A |2\rangle_E \langle 0|_A\langle 1| + \cdots$$

Leading to:

$$\hat{K}_0 = \sqrt{1-p}\, \mathbb{1}, \quad \hat{K}_1 = \sqrt{p}\, \hat{\Pi}_0, \quad \hat{K}_2 = \sqrt{p}\, \hat{\Pi}_1.$$

$$\Rightarrow \mathcal{E}(\hat{\rho}) = \begin{pmatrix} \rho_{00} & (1-p)\rho_{01} \\ (1-p)\rho_{10} & \rho_{11} \end{pmatrix}. \quad (\star)$$

Let's look at the dephasing channel in some detail, as it's a simple but powerful example of open systems and decoherence.

We noted that applying $\mathcal{E}$ N times gave us

$$\mathcal{E}^N(\hat{\rho}) = \begin{pmatrix} \rho_{00} & (1-p)^N \rho_{01} \\ (1-p)^N \rho_{10} & \rho_{11} \end{pmatrix} .$$

Let's think of p as the probability per unit time Δt that the environment acts with our qubit. Write it in terms of a rate Γ, and assume it's small:

$$p = \Gamma \Delta t \ll 1 .$$

Over a longer time

$$t = N \Delta t$$

we have

$$(1-p)^N = (1-\Gamma\Delta t)^N = \left(1 - \frac{\Gamma t}{N}\right)^N \xrightarrow[N\to\infty]{\lim} e^{-\Gamma t}$$

keeping t fixed as $N \to \infty$.

So the density matrix of our qubit becomes

$$\hat{\rho}_A(t) = \begin{pmatrix} \rho_{00} & e^{-\Gamma t}\rho_{01} \\ e^{-\Gamma t}\rho_{10} & \rho_{11} \end{pmatrix},$$

showing the decay to a diagonal form.

Let's think about what this means.

Imagine starting our qubit in an arbitrary pure state $|\phi_0\rangle$,

$$\hat{\rho} = |\phi_0\rangle_A\langle\phi_0| \otimes |0\rangle_E\langle 0|,$$

$$|\phi_0\rangle = \alpha|0\rangle_A + \beta|1\rangle_A.$$

The qubit's density matrix is

$$\hat{\rho}_A(0) = |\phi_0\rangle\langle\phi_0|,$$

$$\rho_{Aij}(0) = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1.$$

Just from staring at the ρ_{ij}'s, it's hard to tell whether $\hat{\rho}$ is pure or mixed.

But we know that if $\hat{\rho}_A$ is pure, it only has one nonzero eigenvector. So it's natural to think that keeping diagonal elements constant & letting off-diagonal ones decay will make a state <u>less</u> pure. And that's true.

Decoherence: evolution of a subsystem from a pure to mixed state, due to becoming entangled with the environment.

Two ways of seeing loss of purity.

First: <u>Entropy</u>. $S[\hat{\rho}] = -\mathrm{Tr}\,\hat{\rho}\log\hat{\rho}$.

For convenience write ρ_{ij} as

$$\rho_{ij} = \begin{pmatrix} a & b \\ b^* & 1-a \end{pmatrix}. \qquad (\mathrm{Tr}=1,\ \rho^+ = \rho.)$$

Hard to take the log of a matrix unless we diagonalize it. (There's always a basis where $\hat{\rho}$ is diagonal.)

Diagonal form of ρ:

$$\rho_{ij} = \begin{pmatrix} \frac{1}{2}(1+x) & 0 \\ 0 & \frac{1}{2}(1-x) \end{pmatrix}, \quad \text{where}$$

$$x = (1-a+a^2+|b|^2)^{1/2} > 0.$$

Skipping some algebra:

$$S[\rho] = -\frac{1}{2}\left\{\log 4 + (1+x)\log(1+x) + (1-x)\log(1-x)\right\}$$

$$\frac{dS}{dx} = -\frac{1}{2}\left[\log(1+x) - \log(1-x)\right] < 0.$$

and $$\frac{dx}{d|b|} = \frac{|b|}{x} > 0.$$

So as off-diagonal term b decreases, x decreases, and therefore S increases.

Decoherence increases entropy.

Not really the Second Law of Thermo, since our qubit is an open system, but similar in spirit.

Second way of seeing dephasing → impurity:

Bloch Sphere.

Recall we can write, for one qubit,

$$\hat{\rho} = \frac{1}{2}(\mathbb{1} + \vec{a}\cdot\vec{\sigma}),$$

where $\vec{a}$ is the "Bloch vector."

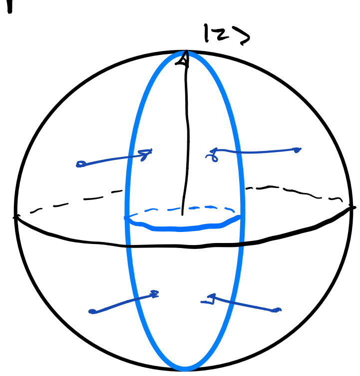
Dephasing (see ⛤) leaves diagonal pieces $(\mathbb{1}, \hat{\sigma}_3)$ invariant, while multiplying off-diagonal pieces $(\hat{\sigma}_1, \hat{\sigma}_2)$ by $(1-p)$.

That's equivalent to transforming the Bloch vector:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \longrightarrow \begin{pmatrix} (1-p)a_1 \\ (1-p)a_2 \\ a_3 \end{pmatrix}.$$

Does not preserve $|\vec{a}| = 1$ (pure states).

In fact, we shrink the Bloch sphere down to a spindle:



Reversibility

Given that decoherence increases entropy, you won't be surprised to learn that it's an irreversible process.

Unitary evolution $\hat{\rho} \to \hat{U}\rho\hat{U}^{+}$ can always be reversed, by $\hat{U}^{-1} = \hat{U}^{+}$. (The Schrödinger equation does not define an arrow of time.)

Quantum channels, however, can be irreversible. Just from the last example, consider the action of dephasing on a density matrix with Bloch vector $\vec{a}_* = \lambda \hat{e}_x$ (vector in the x-direction):

$$\mathcal{E}: \ \vec{a}_* \to (1-p)\,\vec{a}_* .$$

Any reverse map must therefore take

$$\mathcal{E}^{-1}: \ \vec{a}_* \to \frac{1}{1-p}\,\vec{a}_* .$$

But this could yield $|\vec{a}| > 1$. Dephasing (and decoherence more generally) is irreversible.

Footnote: if our subsystem is part of a larger closed system, evolution as a whole is of course reversible (putting aside "collapse of the wave function").

The reason we can't reverse the quantum channel is that the time-reversed evolution would violate some of the assumptions we made along the way — in particular, that A & E started out unentangled.

Indeed, if dim $\mathcal{H}$ = finite for some isolated system, and it evolves for long enough, the system inevitably returns to where it started (to within an arbitrary error ϵ). That's the "Quantum Poincaré Recurrence Theorem." Finite-dim $\mathcal{H}$ is analogous to bounded classical phase space. The recurrence time scales with dim $\mathcal{H}$.

Physical meaning of decoherence

Think about decoherence at the level of wave functions.

$$|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_E$$

$$|\Psi(0)\rangle = (\alpha|\phi_1\rangle + \beta|\phi_2\rangle) \otimes |e_0\rangle$$

$$\rightsquigarrow |\Psi(T)\rangle = \alpha|\phi_1\rangle|e_1\rangle + \beta|\phi_2\rangle|e_2\rangle$$

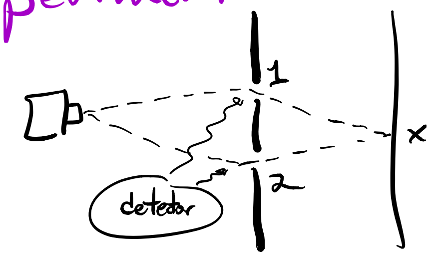
Two important features:

1. increased entanglement b/w A & E.
2. system density matrix diagonalizes in some basis $\Longleftrightarrow$ orthogonal environment states, $\langle e_1|e_2\rangle = 0$.

Not only are $|e_1\rangle$, $|e_2\rangle$ orthogonal, they <u>remain</u> orthogonal for a very long time (though perhaps not strictly forever).

Decoherence matters for understanding interpretations & the measurement problem, but there's a more immediate implication: destruction of quantum interference.

Think of the double-slit experiment.



$$\mathcal{H} = \mathcal{H}_{\text{particle}} \otimes \mathcal{H}_{\text{detector}}$$

Part of state describing particle ending up @ x:

$$(\phi_1 + \phi_2)\,|x\rangle|d_0\rangle$$

Corresponding probability:

$$p(x) = \langle d_0|\langle x|\,(\phi_1+\phi_2)^*(\phi_1+\phi_2)\,|x\rangle|d_0\rangle = |\phi_1|^2 + |\phi_2|^2 + 2\,\mathrm{Re}\,(\phi_1\phi_2^*)$$

$\rightarrow$ possibility of interference.

But if detector peeks at which slit it goes through:

$$\phi_1|x\rangle|d_1\rangle + \phi_2|x\rangle|d_2\rangle, \quad \text{with } \langle d_1|d_2\rangle = 0.$$

$$\therefore\ p(x) = \langle d_1|\langle x|\,\phi_1^*\phi_1\,|x\rangle|d_1\rangle + \langle d_2|\langle x|\,\phi_2^*\phi_2\,|x\rangle|d_2\rangle$$

$$= |\phi_1|^2 + |\phi_2|^2 \quad \rightarrow \text{no interference.}$$

"Measurement" in the two-slit experiment doesn't require an observer — just decoherence.

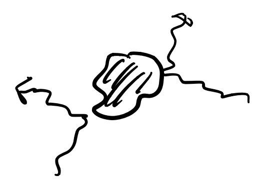
Dephasing as a model of real-world decoherence.

$$|0\rangle_A |0\rangle_E \rightsquigarrow |0\rangle_A |0\rangle_E + |0\rangle_A |1\rangle_E$$

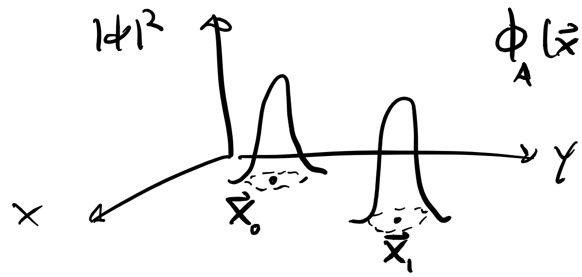
$$|1\rangle_A |0\rangle_E \rightsquigarrow |1\rangle_A |0\rangle_E + |1\rangle_A |2\rangle_E .$$

Qubit states $|0\rangle, |1\rangle$ need not represent actual "spins" — any two-state system will do.

Think of: a rock in outer space, bombarded by 3K CMB photons.



But for some reason, the rock starts in a (pure) state localized sharply around two possible locations in space.



$$\phi_A(\vec{x}) \propto \exp(\vec{x}-\vec{x}_0)^2 + \exp(\vec{x}-\vec{x}_1)^2 .$$

Think of $|0\rangle \sim \int d^3x \, e^{-(\vec{x}-\vec{x}_0)^2} |x\rangle$

and $|1\rangle \sim \int d^3x \, e^{-(\vec{x}-\vec{x}_1)} |x\rangle .$

When does a photon interact with the rock? When it comes into contact (i.e., enters the same spatial location).

Our dephasing channel provides a model for such interactions. Scattered photons evolve to different ($\therefore$ orthogonal) states depending on whether they contact the $|0\rangle$ or $|1\rangle$ parts of the rock wave function.

No danger of repeating or reversing, since there are so many photons.

The decoherence rate Γ (cf. above) is just the inverse time for <u>one</u> photon to scatter. For CMB with $n_\gamma \sim 10^2\ \text{cm}^{-3}$, that's approximately $n_\gamma c \sigma$, where σ is the cross-sectional area. So for a dust particle of radius $r = 10^{-3}$ cm,

$$\Gamma \sim (10^2\ \text{cm}^{-3})\left(10^{10}\ \frac{\text{cm}}{\text{s}}\right)(10^{-6}\ \text{cm}^2) = (10^{-6}\ \text{s})^{-1}.$$

Even tiny dust particles decohere very fast. Large macroscopic objects decohere <u>extremely</u> fast. Earth: $(10^{-22}\,s)^{-1}$!

That's why fighting off decoherence is a challenge for quantum computing.

<u>Schrödinger's Cat</u>: What's the state of the cat before you open the box?

It's <u>not</u> $\frac{1}{\sqrt{2}}\left(|\text{awake}\rangle + |\text{sleep}\rangle\right)$, as you might have thought.

The cat interacts with photons/air/etc. in the box — an environment that quickly decoheres it.

So the cat has a density matrix in the awake/sleep basis

$$\hat{\rho}_{cat} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

long before you open the box.