1 More on the Bloch Sphere (10 points)

In class, we learned to parametrize any pure, qubit state \( |\Psi\rangle \in \mathcal{H} = \mathbb{C}^2 \) in the “standard” basis \( \{|0\rangle, |1\rangle\} \) as follows:

\[
|\Psi\rangle = e^{i\gamma} \left( \cos \left( \frac{\theta}{2} \right) |0\rangle + e^{i\phi} \sin \left( \frac{\theta}{2} \right) |1\rangle \right),
\]

where \( \gamma \) is an overall (and irrelevant) phase and the angles \( \theta \) and \( \phi \) label points on the Bloch Sphere with \( 0 \leq \theta \leq \pi \) and \( 0 \leq \phi \leq 2\pi \).

(a.) Consider a state \( |\Psi\rangle \) parametrized by angles \( (\theta, \phi) \) and the state \( |\tilde{\Psi}\rangle \) geometrically opposite to \( |\Psi\rangle \) on the Bloch sphere. Show that the states \( |\Psi\rangle \) and \( |\tilde{\Psi}\rangle \) are orthogonal. [3 points]

(b.) Any \( 2 \times 2 \) unitary operator \( \hat{U} \) can be written in the form,

\[
\hat{U} = \exp \left( i \left[ \alpha \hat{I} + \beta \vec{u} \cdot \hat{\sigma} \right] \right),
\]

where \( \alpha \) and \( \beta \) are real constants, \( \hat{I} \) is the \( 2 \times 2 \) identity matrix, \( \hat{\sigma} = \{\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\} \) are the Pauli matrices, and \( \vec{u} \) is a unit vector \( (u_x^2 + u_y^2 + u_z^2 = 1) \). (Note that the expression in square brackets is a general Hermitian matrix.) Meanwhile a \( 2 \times 2 \) density operator \( \hat{\rho} \) can be written in the Bloch form

\[
\hat{\rho} = \frac{1}{2} \left( \hat{I} + \vec{a} \cdot \hat{\sigma} \right),
\]

where \( \vec{a} \) is the Bloch vector. Show that the transformation of \( \hat{\rho} \) under \( \hat{U} \) given by \( \hat{U} \hat{\rho} \hat{U}^\dagger \) corresponds to a rotation of the Bloch vector through an angle \( 2\beta \) about the axis \( \vec{u} \). [4 points]

(c.) Compute the operator product \( (\vec{u} \cdot \hat{\sigma})(\vec{v} \cdot \hat{\sigma}) \), where \( \vec{u} \equiv (u_x, u_y, u_z) \) and \( \vec{v} \equiv (v_x, v_y, v_z) \) are (not necessarily unit) vectors. (Try simplifying the result to involve scalar and vector products of \( \vec{u} \) and \( \vec{v} \).) Using this result and your result from (b.), find an expression for \( \hat{\rho}^2 \). [3 points]

2 Purity! (5 points)

A common and popular measure of how mixed a state is, is the so-called Purity measure, which is defined for a state \( \hat{\rho} \) as,

\[
Purity(\hat{\rho}) = \text{Tr} (\hat{\rho}^2)
\]

(a.) Show that a pure state \( \hat{\rho}_{\text{pure}} \) is idempotent \( (\hat{\rho}^2 = \hat{\rho}) \) and hence, find the purity of such a pure state. [1 point]

(b.) The value found above in (a.) is the maximum possible value of purity attainable for a state. What is the minimum value and what is the corresponding density operator? For
concreteness, work in a $d$-dimensional Hilbert space $\mathcal{H}$. \[2 \text{ points}\]

(c.) Let’s return to the expression for a qubit density operator from problem (1) and its square found in part (c.). Calculate the purity of this state as a function of the length of the Bloch vector, and show that $\vec{a} \cdot \vec{a} = 1$ if and only if the state is pure. \[2 \text{ points}\]

3 One Entangled Evening...spent doing Homework (10 points)

Consider a bipartite split of Hilbert space $\mathcal{H}$ into subsystems $\mathcal{A}$ and $\mathcal{B}$ i.e. $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and a (pure) normalized state $|\Psi\rangle \in \mathcal{H}$. Such a state can always be written in the Schmidt decomposition,

$$|\Psi\rangle = \sum_n a_n |\sigma_n\rangle_A \otimes |\lambda_n\rangle_B ,$$

where the states $\{ |\sigma_n\rangle\}$ and $\{ |\lambda_n\rangle\}$ are orthonormal states for $\mathcal{A}$ and $\mathcal{B}$, respectively. (In class we wrote $|\sigma_n\rangle = |\sigma(A)_n\rangle$ and $|\lambda_n\rangle = |\sigma(B)_n\rangle$, but this notation is slightly more compact.)

(a.) A simple quantification of the degree of entanglement between subsystems $\mathcal{A}$ and $\mathcal{B}$ can be done by the quantity $\kappa$ defined as,

$$\kappa = \frac{1}{\sum_n |a_n|^4} .$$

Specialize to the case of $\dim(\mathcal{A}) = \dim(\mathcal{B}) = d$. Find the maximum and minimum values of $\kappa$ and show that the minimum value occurs only if the state is unentangled. \[3 \text{ points}\]

(b.) Using $\kappa$ as the measure (higher the $\kappa$, more entangled the state), show that the Bell states are maximally entangled. \[4 \text{ points}\]

(c.) Consider three qubits, with Hilbert space

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 .$$

A popular state employed and discussed in Quantum Computation and Quantum Information circles is the so-called Werner or $W$- state given by,

$$|W\rangle = \alpha |001\rangle + \beta |010\rangle + \gamma |100\rangle ,$$

where $\alpha$, $\beta$ and $\gamma$ are probability amplitudes, none of which is zero. Compute the reduced density matrices for qubit $A$ and calculate its von Neumann entanglement entropy. \[3 \text{ points}\]