Ph125c Spring 2017
Prof. Sean Carroll
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Homework - 1
Assigned TA: Ashmeet Singh
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Due: 5:00pm, 4/19/2017

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1 More on the Bloch Sphere (10 points)

In class, we learned to parametrize any pure, qubit state $|\Psi\rangle \in \mathcal{H} = \mathbb{C}^2$ in the "standard" basis $\{|0\rangle, |1\rangle\}$ as follows:

$$|\Psi\rangle = e^{i\gamma}\left(\cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle\right) , \tag{1}$$

where γ is an overall (and irrelevant) phase and the angles θ and ϕ label points on the *Bloch Sphere* with $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$.

(a.) Consider a state $|\Psi\rangle$ parametrized by angles (θ, ϕ) and the state $|\tilde{\Psi}\rangle$ geometrically opposite to $|\Psi\rangle$ on the Bloch sphere. Show that the states $|\Psi\rangle$ and $|\tilde{\Psi}\rangle$ are orthogonal. [3 points]

(b.) Any 2×2 unitary operator $\hat{U}$ can be written in the form,

$$\hat{U} = \exp\left(i\left[\alpha\hat{\mathbb{I}} + \beta\vec{u}\cdot\hat{\vec{\sigma}}\right]\right) , \tag{2}$$

where α and β are real constants, $\hat{\mathbb{I}}$ is the 2×2 identity matrix, $\hat{\vec{\sigma}} = \{\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\}$ are the Pauli matrices, and $\vec{u}$ is a unit vector ($u_x^2 + u_y^2 + u_z^2 = 1$). (Note that the expression in square brackets is a general Hermitian matrix.) Meanwhile a 2×2 density operator $\hat{\rho}$ can be written in the Bloch form

$$\hat{\rho} = \frac{1}{2}\left(\hat{\mathbb{I}} + \vec{a}\cdot\hat{\vec{\sigma}}\right) , \tag{3}$$

where $\vec{a}$ is the Bloch vector. Show that the transformation of $\hat{\rho}$ under $\hat{U}$ given by $\hat{U}\hat{\rho}\hat{U}^\dagger$ corresponds to a rotation of the Bloch vector through an angle 2β about the axis $\vec{u}$. [4 points]

(c.) Compute the operator product $(\vec{u}\cdot\vec{\sigma})(\vec{v}\cdot\vec{\sigma})$, where $\vec{u} \equiv (u_x, u_y, u_z)$ and $\vec{v} \equiv (v_x, v_y, v_z)$ are (not necessarily unit) vectors. (Try simplifying the result to involve scalar and vector products of $\vec{u}$ and $\vec{v}$.) Using this result and your result from (b.), find an expression for $\hat{\rho}^2$. [3 points]

2 Purity! (5 points)

A common and popular measure of how mixed a state is, is the so-called *Purity* measure, which is defined for a state $\hat{\rho}$ as,

$$\text{Purity}(\hat{\rho}) = \text{Tr}\left(\hat{\rho}^2\right) \tag{4}$$

(a.) Show that a pure state $\hat{\rho}_{\text{pure}}$ is idempotent ($\hat{\rho}^2 = \hat{\rho}$) and hence, find the purity of such a pure state. [1 point]

(b.) The value found above in (a.) is the maximum possible value of purity attainable for a state. What is the minimum value and what is the corresponding density operator? For concreteness, work in a d-dimensional Hilbert space $\mathcal{H}$. [2 points]

(c.) Let's return to the expression for a qubit density operator from problem (1) and its square found in part (c.). Calculate the purity of this state as a function of the length of the Bloch vector, and show that $\vec{a} \cdot \vec{a} = 1$ if and only if the state is pure. [2 points]

3 One Entangled Evening...spent doing Homework (10 points)

Consider a bipartite split of Hilbert space $\mathcal{H}$ into subsystems $\mathcal{A}$ and $\mathcal{B}$ *i.e.* $\mathcal{H} = \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$ and a (pure) normalized state $|\Psi\rangle \in \mathcal{H}$. Such a state can always be written in the Schmidt decomposition,

$$|\Psi\rangle = \sum_n a_n |\sigma_n\rangle_A \otimes |\lambda_n\rangle_B \ , \tag{5}$$

where the states $\{|\sigma_n\rangle\}$ and $\{|\lambda_n\rangle\}$ are orthonormal states for $\mathcal{A}$ and $\mathcal{B}$, respectively. (In class we wrote $|\sigma_n\rangle = |\sigma_n^{(A)}\rangle$ and $|\lambda_n\rangle = |\sigma_n^{(B)}\rangle$, but this notation is slightly more compact.)

(a.) A simple quantification of the degree of entanglement between subsystems $\mathcal{A}$ and $\mathcal{B}$ can be done by the quantity κ defined as,

$$\kappa = \frac{1}{\sum_n |a_n|^4} \ . \tag{6}$$

Specialize to the case of $\dim(\mathcal{A}) = \dim(\mathcal{B}) = d$. Find the maximum and minimum values of κ and show that the minimum value occurs only if the state is unentangled. [3 points]

(b.) Using κ as the measure (higher the κ, more entangled the state), show that the Bell states are maximally entangled. [4 points]

(c.) Consider three qubits, with Hilbert space

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2. \tag{7}$$

A popular state employed and discussed in Quantum Computation and Quantum Information circles is the so-called *Werner* or W- state given by,

$$|W\rangle = \alpha\,|001\rangle + \beta\,|010\rangle + \gamma\,|100\rangle \ , \tag{8}$$

where α, β and γ are probability amplitudes, none of which is zero. Compute the reduced density matrices for qubit A and calculate its von Neumann entanglement entropy. [3 points]
