Ph125c Spring 2017
Prof. Sean Carroll
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Homework - 2
Assigned TA: Ashmeet Singh
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Due: 5:00pm, 4/26/2017

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1 Traces, Traces Everywhere (5 points)

(a.) [2 points] Consider a system described at $t = 0$ by the density operator $\hat{\rho}(0) \in \mathbb{L}(\mathcal{H})$ for a finite dimensional Hilbert space $\mathcal{H}$. Time evolution of the system is governed by the unitary operator $\hat{U}(t)$. Express Tr $(\hat{\rho}^n(t))$ for the time-evolved density operator in terms of $\hat{\rho}(0)$.

(b.) [3 points] Not all quantities Tr $(\hat{\rho}^n)$ are independent. To illustrate this idea, consider a general qubit state $\hat{\rho}$ and write the following quantities as functions of $\mathrm{Tr}\,(\hat{\rho}) = 1$ and $\mathrm{Tr}\,(\hat{\rho}^2)$: (i.) $\mathrm{Tr}\,(\hat{\rho}^3)$ and, (ii.) $\det(\hat{\rho})$, the determinant of the 2×2 matrix associated with ρ.

With some work, one can prove that for a d-dimensional state space, only the first d values of n in Tr $(\hat{\rho}^n)$ yield independent quantities.

2 More on von Neumann Entropy (5 points)

As discussed in class, the von Neumann entropy of a general (possibly mixed) density operator $\hat{\rho} \in \mathbb{L}(\mathcal{H})$ for a finite-dimensional Hilbert space $\mathcal{H}$ is given by,

$$S(\hat{\rho}) = -\mathrm{Tr}\,(\hat{\rho} \log \hat{\rho}) \ . \tag{1}$$

In this question, we will explore various properties and features of this measure of entanglement/mixedness.

(a.) [1 point] In a statement or two, argue why the von Neumann entropy is invariant under a unitary transformation $\hat{U}$, that is,

$$S(\hat{U}\hat{\rho}\hat{U}^\dagger) = S(\hat{\rho}) \ . \tag{2}$$

(b.) [2 points] If we form a linear combination of two density operators then the result tends to be a more mixed state than that associated with either of the component density operators. Show that if $\hat{\rho} = p_1\hat{\rho}_1 + p_2\hat{\rho}_2$, where $\hat{\rho}_1$ and $\hat{\rho}_2$ are density operators and p_1 and p_2 are probabilities ($p_1 + p_2 = 1$), then,

$$S(\hat{\rho}) \geq p_1 S(\hat{\rho}_1) + p_2 S(\hat{\rho}_2) \ , \tag{3}$$

with the equality holding only if p_1 or p_2 is zero or if $\hat{\rho}_1 = \hat{\rho}_2$. This property is called ***concavity*** and also generalizes to more than two components (you don't need to prove this though),

$$S\left(\sum_i p_i \hat{\rho}_i\right) \geq \sum_i p_i S(\hat{\rho}_i) \ , \quad \text{with} \quad \sum_i p_i = 1 \tag{4}$$

(c.) [2 points] Now, consider a bipartite split of the Hilbert space $\mathcal{H} = \mathcal{A} \otimes \mathcal{B}$. Show that if the two subsystems are statistically independent, so that $\hat{\rho}_{AB} = \hat{\rho}_A \otimes \hat{\rho}_B$, then the von Neumann entropy is *additive*, $S(\hat{\rho}_{AB}) = S(\hat{\rho}_A) + S(\hat{\rho}_B)$.

Hint: Think how eigenvalues of $\hat{\rho}_A \otimes \hat{\rho}_B$ connect with those of $\hat{\rho}_A$ and $\hat{\rho}_B$.

3 Let's measure a GHZ! (5 points)

For a single qubit, let us define the set of eigenstates of the Pauli operators as follows,

$$|+z\rangle = |0\rangle \quad (5)$$

$$|-z\rangle = |1\rangle \quad (6)$$

$$|+x\rangle = \tfrac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \quad (7)$$

$$|-x\rangle = \tfrac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \quad (8)$$

$$|+y\rangle = \tfrac{1}{\sqrt{2}} (|0\rangle + i\,|1\rangle) \quad (9)$$

$$|-y\rangle = \tfrac{1}{\sqrt{2}} (|0\rangle - i\,|1\rangle) \quad (10)$$

Now consider the three qubit Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. One of the most popular states used in quantum information is the so-called *GHZ* (Greenberger - Horne - Zeilinger) state,

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle) \ . \quad (11)$$

This state displays many non-classical features, but that is for later in the term. For now, suppose we now measure the first qubit in the $\{|\pm x\rangle\}$ basis.

(a.) [1 point] Construct the projection operators corresponding to this measurement and show they satisfy the requirement of being a *Projection-Valued-Measure (PVM)*.

(b.) [2 points] Write down the pre-measurement density operator of the GHZ state, call it $\hat{\rho}_0$. Now say we perform the measurement but we *do not know* the measurement outcome yet. Compute this post-measurement density operator.

(c.) [2 points] In this other case, we measure and also observe the outcome of the measurement. What is the post-measurement density operators in the two possible outcomes and their respective probabilities? Also, explicitly write down the residual state of the second and third qubits in each of these two cases.

4 Thermal Spins in B-field (5 points)

Consider a spin-1/2 particle in a canonical ensemble at a constant temperature T, with the operator corresponding to its magnetic moment $\hat{\vec{\mu}} = \gamma \hat{\vec{S}}$, where $\vec{S} = \hat{\vec{\sigma}}/2$ is its spin and γ its gyromagnetic ratio. It is subjected to a uniform magnetic field in the z direction, $\vec{B} = B\hat{z}$ and thus, the Hamiltonian is simply

$$\hat{H} = -\hat{\vec{\mu}} \cdot \vec{B}. \quad (12)$$

(a.) [2 points] Write down the thermal density matrix for this canonical ensemble in the standard $|0\rangle = |+z\rangle, |1\rangle = |-z\rangle$ basis.

(b.) [2 points] From this thermal density matrix, compute the ensemble average for the three components of spin of one of the particles, *i.e.* $\langle\hat{\sigma}_x\rangle$, $\langle\hat{\sigma}_y\rangle$ and $\langle\hat{\sigma}_z\rangle$.

(c.) [1 point] Now consider there are N such spin-1/2 particles per unit volume. Such an ensemble will have a net magnetization (net magnetic moment per volume) $\vec{M} = \gamma\left\langle\hat{\vec{\sigma}}\right\rangle/2$. Consider the high temperature limit of this system. Calculate the net magnetization of the ensemble to lowest non-trivial order and verify it satisfies Curie's Law with its characteristic $1/T$ dependence.
