Ph125c Spring 2017 **Homework - 3**
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1 More General von Neumann Measurements [10 Points]

Recall that the Bloch sphere representations of orthogonal qubit states whose Bloch vectors are the antipodal pair $\pm\hat{n}$ are $\rho_{\pm n} = |\pm\hat{n}\rangle\langle\pm\hat{n}| = \frac{1}{2}(\mathbb{I} \pm \hat{n}\cdot\vec{\sigma})$. We have seen in class how, using the von Neumann model, we can perform the PVM $\{|+\hat{n}\rangle\langle+\hat{n}|, |-\hat{n}\rangle\langle-\hat{n}|\}$ by letting the system interact with a single ancilla system (representing the measurement apparatus) using the interaction $H_I = g(\hat{n}\cdot\vec{\sigma})\otimes\sigma_x$.

This von Neumann model of measurement is useful for many reasons. One is that it allows us to measure a quantum system non-destructively (for example, allowing us to measure a photon without absorbing it into a photodetector). Recall that for a qubit, one can not simultaneously know the system's eigenvalues for more than one of the Pauli matrices, because they do not commute. Thus, for example, it would seem that σ_x and σ_z can not be measure simultaneously. But using the von Neumann measurement model, it would seem that maybe we can! What if we instead let our system interact with *two* ancilla systems representing *two* measurement apparati, where the two observables that are measured are non-commuting? For example, we could consider the following interaction Hamiltonian between the system S that we want to measure, and the two ancillary measurement systems $M1$ and $M2$:

$$H_I = g\left(\sigma_z\otimes\sigma_x\otimes\mathbb{I} + \sigma_x\otimes\mathbb{I}\otimes\sigma_x\right). \tag{1}$$

Suppose that we let the joint system $S\otimes M1\otimes M2$ evolve under this Hamiltonian (with the corresponding evolution operator U) for time $t = \frac{\pi}{4g}$. If we then measure some PVM $\mathcal{M} = \{M_{00}, M_{01}, M_{10}, M_{11}\}$ (where the subscripts represent the basis states in $M1\otimes M2$). This will correspond to doing a simultaneous joint measurement of σ_x and σ_z on the system S, each of which gives *incomplete information* about the observables σ_x and σ_z. Our goal here will be to understand how this works using POVMs on the single qubit system. Note that in this problem, all systems are qubits: $M1\cong M2\cong S\cong\mathbb{C}^2$.

(a.) [2 points] Denote the initial state of system S, which is initially uncorrelated from $M1$ and $M2$, as $|\psi\rangle_S$. Assume that $M1$ and $M2$ are each initially in their ready states $|0\rangle_{1/2}\langle 0|_{1/2}$. Using the generalized Born rule for calculating probabilities of measurement outcomes (recall, it is implemented via a trace over all three subsystems), find an expression for the POVM elements $\{E_{ij}\}$ that should correspond to the desired measurement on S as a partial trace over subsystems $M1$ and $M2$. This partial trace should involve the interaction evolution operator U discussed above, and the PVM elements $\{M_{00}, M_{01}, M_{10}, M_{11}\}$.

Now, consider the two orthogonal Bloch sphere axes (careful: the pure states in these directions are not orthogonal!), $\hat{u} = \frac{1}{\sqrt{2}}(\hat{x}+\hat{z})$ and $\hat{v} = \frac{1}{\sqrt{2}}(-\hat{x}+\hat{z})$. Consider the observables $\mathcal{U} = \hat{u}\cdot\vec{\sigma}$ and $\mathcal{V} = \hat{v}\cdot\vec{\sigma}$, with corresponding eigenvectors $|\pm\hat{u}/\hat{v}\rangle$. If $|\pm\rangle$ are the eigenvectors of σ_x, then it is possible to show (*please don't*) that the evolution operator U discussed above can be written

$$
\begin{aligned}
U = \quad & |{+u}\rangle\langle{+u}| \otimes (|{++}\rangle\langle{++}| + i\,|{--}\rangle\langle{--}|) \\
& + i\,|{-u}\rangle\langle{-u}| \otimes (|{++}\rangle\langle{++}| - i\,|{--}\rangle\langle{--}|) \\
& + |{+v}\rangle\langle{+v}| \otimes (|{+-}\rangle\langle{+-}| + i\,|{-+}\rangle\langle{-+}|) \\
& + i\,|{-v}\rangle\langle{-v}| \otimes (|{+-}\rangle\langle{+-}| - i\,|{-+}\rangle\langle{-+}|). \qquad (2)
\end{aligned}
$$

Let the PVM operators $\{M_{ij}\}$ discussed above be the projectors onto the states $|\phi_{ij}\rangle$, where

$$
\begin{aligned}
|\phi_{00}\rangle &= \tfrac{1}{\sqrt{2}}(|{++}\rangle + i\,|{--}\rangle) \\
|\phi_{01}\rangle &= \tfrac{1}{\sqrt{2}}(|{+-}\rangle + i\,|{-+}\rangle) \\
|\phi_{10}\rangle &= \tfrac{1}{\sqrt{2}}(|{+-}\rangle - i\,|{-+}\rangle) \\
|\phi_{11}\rangle &= \tfrac{1}{\sqrt{2}}(|{++}\rangle - i\,|{--}\rangle). \qquad (3)
\end{aligned}
$$

(b.) [2 points] Calculate the corresponding POVM elements E_{ij} as defined in part a, and show that they indeed form a POVM (that is, they are positive operators which sum to the identity). (Hint: think about what density operator you get when you add antipodal points on the Bloch sphere).

Okay, now we will consider the description of a new measurement based on the measurement just described above. The new measurement procedure is as as follows: perform the joint measurement procedure above, which returns outcome (i, j), and *ignore* the outcome j of measurement apparatus $M2$. Basic probability tells us that $\Pr(i) = \sum_j \Pr(i|j)$, where Pr(i|j) is the conditional probability of i, given that j is true. Our new POVM $\mathcal{F}$ will have elements $\{F_i\}$, $i \in \{0, 1\}$.

(c.) [2 Points] Write down an expression for $\Pr(i)$ for an arbitrary system state $|\chi\rangle_S$ as a single trace using the POVM elements E_{ij}. Deduce from this expressions for the operators F_i in terms of the operators E_{ij}.

(d.) [2 Points] Show that we can also write $F_i = q\,|i\rangle\langle i| + (1-q)\frac{\mathbb{I}}{2}$ for some appropriate value of q (which you'll find in the process), and show that the operators $\{F_i\}$ form a POVM.

(e.) [1 Point] Give an *interpretation* for the measurement procedure that POVM $\{F_i\}$ represents that allows us to understand it as a *noisy* measurement of σ_z. This interpretation should be something like 'with probability p, a measurement of *insert measurement here* is performed, and with probability $1-p$, the POVM *insert POVM here* is measured'. (Hint: the POVM should be able to be interpreted as just ignoring the system and flipping a fair coin.)

(f.) [1 Point] Now, go back to the definition of $\mathcal{F}$ given before part (c.), and instead consider the case that when we get measurement outcome (i, j) we instead ignore the outcome i of measurement apparatus $M1$ and keep j, calling this POVM $\mathcal{G} = \{G_j\}$. You need not show all of the work for parts (c.), (d.), and (e.) again, but give an analogous interpretation to that given in part (e.) but for $\mathcal{G}$. (It should just be a difference in the observable measured with probability p.)

One can now conclude that the POVM $\{E_{ij}\}$ is a *noisy joint measurement* of σ_x and σ_z on the *single qubit system* S that gives *incomplete information* about about the two incompatible observables.

2 Kraus [5 Points]

Quantum information theory uses the CNOT or controlled-not gate, a unitary operation on two qubits that acts as

$$\begin{aligned} U_{\text{cnot}}: \quad & |00\rangle \to |00\rangle & (4)\\ & |01\rangle \to |01\rangle & (5)\\ & |10\rangle \to |11\rangle & (6)\\ & |11\rangle \to |10\rangle\,. & (7) \end{aligned}$$

In other words, the first ("control") qubit stays fixed, while the second ("target") qubit flips when the first is a 1, and stays fixed when the first is a 0.

(a.) [1 Point] Find the Kraus operators for the quantum channel $\mathcal{E}_{CNOT}$, where the system qubit is the control qubit, and the environment is the target qubit. Assume that the target/environment qubit starts in state $|0\rangle\langle 0|$

One important example of a quantum channel is the depolarizing channel. In general, the depolarizing channel on a qudit with Hilbert space $\mathbb{C}^d$ can be written

$$\mathcal{E}(\rho) = (1-p)\rho + p\frac{\mathbb{I}}{d}, \tag{8}$$

giving the interpretation that with probability p, the input state is replaced by a completely unknown random state (which we must represent with the maximally mixed state $\frac{\mathbb{I}}{d}$), and otherwise does nothing.

(b.) [2 Points] Consider the depolarizing channel for a qubit ($d = 2$). Find a set of Kraus operators for this channel. (Hint: Consider the decomposition of the qubit density operator into Pauli operators via the Bloch sphere. What does conjugating a Pauli operator by another Pauli operator do? Using this information, you should be able to write the identity operator as a linear combination of an arbitrary density operator, plus conjugations by Paulis.)

(c.) [1 Point] By studying the action of the depolarizing channel on the Bloch sphere representation of a qubit, describe what this channel does to the Bloch sphere.

(d.) [1 Point] While the interpretation given makes it seem like p should be between 0 and 1, this actually need not be true for the formula given for the depolarizing channel to correctly represent a quantum channel. Find the most general bounds on p which allow for $\mathcal{E}(\rho) = (1-p)\rho + p\frac{\mathbb{I}}{d}$ to be a quantum channel for $d = 2$.

3 Stinespring and Kraus [5 points]

It was discussed in class that the unitary evolution of a density matrix is not the most general way a quantum system can evolve, but rather a *quantum channel* represented by the CPTP map $\mathcal{E} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ is. We explored the Kraus representation of such a quantum channel and saw that this kind of map bears a similarity to a mixture of unitary evolutions (that is, a density matrix ρ can in general evolve to become a mixture of density matrices under a quantum channel). It was demonstrated in class that in appropriate situations, Kraus operators describing the evolution of a subsystem can be obtained by considering unitary evolution of a larger system, and then tracing out everything except the subsystem of interest (via the partial trace).

Kraus operators do not always need to be obtained in this fashion, but there is a theorem from operator theory called the Stinespring Dilation Theorem that says you always can if you want to. In our language, the Stinespring Dilation Theorem says: *Any CPTP map* $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A)$ *can be expressed as the reduced action of a single unitary operator acting on an extended Hilbert space* $U : \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_A \otimes \mathcal{H}_B$. That is, we can write

$$\mathcal{E}(\rho) = \mathrm{Tr}_B(U\rho \otimes \sigma U^\dagger), \tag{9}$$

where σ is some state on an environment Hilbert space $\mathcal{H}_B$ of unspecified dimension.

Show that this is true for a quantum channel $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A)$ on a d-dimensional quantum system $\mathcal{H}_A = \mathbb{C}^d$, by explicitly constructing the unitary U in terms of the Kraus operators $\{K_i\}_{i=0}^{k-1}$ of $\mathcal{E}$, assuming initial environment state $\sigma = |0\rangle\langle 0|$. (Hint: Similarly to how there is a unitary freedom in choosing Kraus operators to implement a quantum channel, the unitary matrix U here is not unique. In fact, you should be able to argue that only d columns of U are fixed by the Kraus operators. If you choose to reverse the ordering of your Hilbert spaces (and hence their bases) to $\mathcal{H}_B \otimes \mathcal{H}_A$ instead of $\mathcal{H}_A \otimes \mathcal{H}_B$, it will be exactly the first d columns of $U : \mathcal{H}_B \otimes \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_A$ that will be fixed by your Kraus operators, and that part of your unitary will have a nice block form. You can choose to work in this reversed basis for convenience if you wish. The remaining columns of U can be chosen arbitrarily as long as all of the columns end up orthonormal).
