Ph125c Spring 2017 **Homework - 3 Solutions**
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1 More General von Neumann Measurements [10 Points]

Recall that the Bloch sphere representations of orthogonal qubit states whose Bloch vectors are the antipodal pair $\pm\hat{n}$ are $\rho_{\pm n} = |\pm\hat{n}\rangle\langle\pm\hat{n}| = \frac{1}{2}(\mathbb{I} \pm \hat{n}\cdot\vec{\sigma})$. We have seen in class how, using the von Neumann model, we can perform the PVM $\{|+\hat{n}\rangle\langle+\hat{n}|, |-\hat{n}\rangle\langle-\hat{n}|\}$ by letting the system interact with a single ancilla system (representing the measurement apparatus) using the interaction $H_I = g(\hat{n}\cdot\vec{\sigma})\otimes\sigma_x$.
This von Neumann model of measurement is useful for many reasons. One is that it allows us to measure a quantum system non-destructively (for example, allowing us to measure a photon without absorbing it into a photodetector). Recall that for a qubit, one can not simultaneously know the system's eigenvalues for more than one of the Pauli matrices, because they do not commute. Thus, for example, it would seem that σ_x and σ_z can not be measure simultaneously. But using the von Neumann measurement model, it would seem that maybe we can! What if we instead let our system interact with *two* ancilla systems representing *two* measurement apparati, where the two observables that are measured are non-commuting? For example, we could consider the following interaction Hamiltonian between the system S that we want to measure, and the two ancillary measurement systems $M1$ and $M2$:

$$H_I = g\left(\sigma_z\otimes\sigma_x\otimes\mathbb{I} + \sigma_x\otimes\mathbb{I}\otimes\sigma_x\right). \tag{1}$$

Suppose that we let the joint system $S\otimes M1\otimes M2$ evolve under this Hamiltonian (with the corresponding evolution operator U) for time $t = \frac{\pi}{4g}$. If we then measure some PVM $\mathcal{M} = \{M_{00}, M_{01}, M_{10}, M_{11}\}$ (where the subscripts represent the basis states in $M1\otimes M2$). This will correspond to doing a simultaneous joint measurement of σ_x and σ_z on the system S, each of which gives *incomplete information* about the observables σ_x and σ_z. Our goal here will be to understand how this works using POVMs on the single qubit system. Note that in this problem, all systems are qubits: $M1 \cong M2 \cong S \cong \mathbb{C}^2$.

(a.) [2 points] Denote the initial state of system S, which is initially uncorrelated from $M1$ and $M2$, as $|\psi\rangle_S$. Assume that $M1$ and $M2$ are each initially in their ready states $|0\rangle_{1/2}\langle 0|_{1/2}$. Using the generalized Born rule for calculating probabilities of measurement outcomes (recall, it is implemented via a trace over all three subsystems), find an expression for the POVM elements $\{E_{ij}\}$ that should correspond to the desired measurement on S as a partial trace over subsystems $M1$ and $M2$. This partial trace should involve the interaction evolution operator U discussed above, and the PVM elements $\{M_{00}, M_{01}, M_{10}, M_{11}\}$.

Solution: The Born rule tells is that if we have a POVM (which includes the special case of a PVM) $\{E_a\}_a$ on Hilbert space $\mathcal{H}$, then the probability of obtaining measurement outcome a when you measure this POVM on state $\rho\in\mathcal{L}(\mathcal{H})$ is $\Pr(a) = \mathrm{Tr}(\rho E_a)$. The state we would like to simultaneously measure σ_z and σ_x on is the state $\rho_S = |\psi\rangle\langle\psi|_S \in \mathcal{L}(S)$. We are imagining two ways of performing this measurement: as an indirect measurement (von Neumann) on an auxiliary system, or as a direct measurement on the system itself. The first (indirect/von Neumann) way is to couple the system of interest S to two distinct measurement devices with Hilbert spaces $M1$ and $M2$ respectively. To do this, we take the initially uncorrelated state on the entire system, $\rho = |\psi\rangle\langle\psi|_S \otimes |0\rangle\langle 0|_{M1} \otimes |0\rangle\langle 0|_{M2}$, evolve the system with interaction unitary U that should entangle them, giving state $U\rho U^\dagger$, and then measuring the PVM $\{M_{ij}\}_{ij}$ on systems $M1$ and $M2$. Because the state that you're measuring is a state on $S \otimes M1 \otimes M2$, your PVM operators should also be operators on $S \otimes M1 \otimes M2$, and we can accomplish this by tensoring on an identity operator for system S (because the measurement operators are acting on $M1 \otimes M2$ and not touching S). So, our PVM that we will measure on $U\rho U^\dagger$ is $\{I_S \otimes M_{ij}\}_{ij}$, and the outcome ij will be obtained with probability given by the Born rule:

$$\Pr(ij) = \mathrm{Tr}\left(U\rho U^\dagger I_S \otimes M_{ij}\right). \tag{2}$$

Now, the whole point is that because our state on S has become entangled with the state on $M1 \otimes M2$ via U, each different measurement outcome ij for our measurement performed on $M1 \otimes M2$ gives us different information about the state of the system S. To understand what this information is, we will find the equivalent POVM $\{E_{ij}\}_{ij}$ that we could just measure on $\rho_S \in \mathcal{L}(S)$ that would give us the same information. For the measurement on $M1 \otimes M2$ to be equivalent to a measurement on S, the outcome statistics for each outcome ij should match. That is, we should demand that if measuring PVM $\{I_S \otimes M_{ij}\}_{ij}$ on state $U\rho U^\dagger$ gives us the same information about ρ_S as measuring $\{E_{ij}\}_{ij}$ on ρ_S, then we must have, by the Born rule,

$$\Pr(ij) = \mathrm{Tr}(\rho_S E_{ij}) = \mathrm{Tr}(U\rho U^\dagger I_S \otimes M_{ij}). \tag{3}$$

Note that the first trace is a trace over Hilbert space S because the argument of the trace is a linear operator on S, whereas the second trace is a trace over Hilbert space $S \otimes M1 \otimes M2$ because its argument is a linear operator on $S \otimes M1 \otimes M2$. The goal here will be to work the second trace into a form that looks like the first trace, allowing us to identify a formula for E_{ij} in terms of U and M_{ij}.

The first thing that we can do is use the cyclic property of the trace to move one of the Us in the second trace, and replace ρ with its explicit expression:

$$\mathrm{Tr}(U\rho U^\dagger I_S \otimes M_{ij}) = \mathrm{Tr}(\rho_S \otimes |0\rangle\langle 0|_{M1} \otimes |0\rangle\langle 0|_{M2} U^\dagger I_S \otimes M_{ij} U) \tag{4}$$

The trace over a composite Hilbert space $S \otimes M1 \otimes M2$ can be implemented as separate partial traces over each of S, $M1$, and $M2$ separately, in any order you would like. Taking the partial trace over $M1 \otimes M2$ and leaving the trace over S incomplete, one finds

$$\mathrm{Tr}_{S\otimes M1\otimes M2}(\rho_S \otimes |0\rangle\langle 0|_{M1} \otimes |0\rangle\langle 0|_{M2} U^\dagger I_S \otimes M_{ij} U) = \mathrm{Tr}_S(\rho_S \mathrm{Tr}_{M1\otimes M2}(I_S \otimes |0\rangle\langle 0|_{M1} \otimes |0\rangle\langle 0|_{M2} U^\dagger I_S \otimes M_{ij} U)). \tag{5}$$

We can now compare this to the POVM Born rule for outcome ij and find that

$$\mathrm{Tr}_S(\rho_S \mathrm{Tr}_{M1\otimes M2}(I_S \otimes |0\rangle\langle 0|_{M1} \otimes |0\rangle\langle 0|_{M2} U^\dagger I_S \otimes M_{ij} U)) = \mathrm{Tr}_S(\rho_S E_{ij}). \tag{6}$$

Hence, we can identify that the POVM

$$\{E_{ij}\}_{ij} = \{\mathrm{Tr}_{M1\otimes M2}(I_S \otimes |0\rangle\langle 0|_{M1} \otimes |0\rangle\langle 0|_{M2} U^\dagger I_S \otimes M_{ij} U)\}_{ij}, \tag{7}$$

which is a measurement implemented directly on state ρ_S, is equivalent to the PVM $\{I_S \otimes M_{ij}\}_{ij}$ implemented indirectly by measuring the state $U\rho U^\dagger$.

Now, consider the two orthogonal Bloch sphere axes (careful: the pure states in these directions are not orthogonal!), $\hat{u} = \frac{1}{\sqrt{2}}(\hat{x} + \hat{z})$ and $\hat{v} = \frac{1}{\sqrt{2}}(-\hat{x} + \hat{z})$. Consider the observables $\mathcal{U} = \hat{u} \cdot \vec{\sigma}$ and $\mathcal{V} = \hat{v} \cdot \vec{\sigma}$, with corresponding eigenvectors $|\pm\hat{u}/\hat{v}\rangle$. If $|\pm\rangle$ are the eigenvectors of σ_x, then it is possible to show (*please don't*) that the evolution operator U discussed above can be written

$$\begin{aligned} U = \quad & |+u\rangle \langle +u| \otimes (|++\rangle \langle ++| + i\,|--\rangle \langle --|) \\ & + i\,|-u\rangle \langle -u| \otimes (|++\rangle \langle ++| - i\,|--\rangle \langle --|) \\ & + |+v\rangle \langle +v| \otimes (|+-\rangle \langle +-| + i\,|-+\rangle \langle -+|) \\ & + i\,|-v\rangle \langle -v| \otimes (|+-\rangle \langle +-| - i\,|-+\rangle \langle -+|). \end{aligned} \tag{8}$$

Let the PVM operators $\{M_{ij}\}$ discussed above be the projectors onto the states $|\phi_{ij}\rangle$, where

$$\begin{aligned} |\phi_{00}\rangle &= \tfrac{1}{\sqrt{2}}(|++\rangle + i\,|--\rangle) \\ |\phi_{01}\rangle &= \tfrac{1}{\sqrt{2}}(|+-\rangle + i\,|-+\rangle) \\ |\phi_{10}\rangle &= \tfrac{1}{\sqrt{2}}(|+-\rangle - i\,|-+\rangle) \\ |\phi_{11}\rangle &= \tfrac{1}{\sqrt{2}}(|++\rangle - i\,|--\rangle). \end{aligned} \tag{9}$$

(b.) [2 points] Calculate the corresponding POVM elements E_{ij} as defined in part a, and show that they indeed form a POVM (that is, they are positive operators which sum to the identity). (Hint: think about what density operator you get when you add antipodal points on the Bloch sphere).

Solution: We would now like to plug in our explicit expressions for U and M_{ij} in to our result from part (a.) to explicitly calculate the operators E_{ij}. Because U is such an unwieldy looking operator, this seems like a lot of work. But there are some observations one can make to drastically simplify the calculation. First, notice that because of the projector $I_S \otimes |0\rangle \langle 0|_{M1} \otimes |0\rangle \langle 0|_{M2}$ sitting on the left hand side of the trace, only the part of the trace where you sandwich your operators with $I_S \otimes \langle 0|_{M1} \otimes \langle 0|_{M2}$ and $I_S \otimes |0\rangle_{M1} \otimes |0\rangle_{M2}$ will be non-zero, because all otherse will give orthogonal inner products on this projector. So,

$$E_{ij} = (I_S \otimes \langle 0|_{M1} \otimes \langle 0|_{M2})\, I_S \otimes |0\rangle \langle 0|_{M1} \otimes |0\rangle \langle 0|_{M2}\, U^\dagger I_S \otimes M_{ij} U\, (I_S \otimes |0\rangle_{M1} \otimes |0\rangle_{M2}) \tag{10}$$

$$= (I_S \otimes \langle 0|_{M1} \otimes \langle 0|_{M2})\, U^\dagger I_S \otimes M_{ij} U\, (I_S \otimes |0\rangle_{M1} \otimes |0\rangle_{M2}) \tag{11}$$

$$\tag{12}$$

Now, looking at just the part $U\,(I_S \otimes |0\rangle_{M1} \otimes |0\rangle_{M2})$, we see that we will need to calculate the inner products $\langle ++|\,00\rangle$, $\langle +-|\,00\rangle$, $\langle -+|\,00\rangle$, and $\langle --|\,00\rangle$, which are all equal to $\frac{1}{2}$. So, we can see that

$$\begin{aligned} U\,(I_S \otimes |0\rangle_{M1} \otimes |0\rangle_{M2}) = \quad & \tfrac{1}{2}\ \ |+u\rangle \langle +u| \otimes (|++\rangle + i\,|--\rangle) \langle 00| \\ & + \tfrac{1}{2} i\,|-u\rangle \langle -u| \otimes (|++\rangle - i\,|--\rangle) \langle 00| \\ & + \tfrac{1}{2}\,|+v\rangle \langle +v| \otimes (|+-\rangle + i\,|-+\rangle) \langle 00| \\ & + \tfrac{1}{2} i\,|-v\rangle \langle -v| \otimes (|+-\rangle - i\,|-+\rangle) \langle 00|\,. \end{aligned} \tag{13}$$

or

$$U\,(I_S \otimes |0\rangle_{M1} \otimes |0\rangle_{M2}) = \quad \tfrac{1}{\sqrt{2}}\ \ |+u\rangle \langle +u| \otimes |\phi_{00}\rangle \langle 00|$$

$$
\begin{aligned}
&+ \tfrac{1}{\sqrt{2}} i \left|-u\right\rangle \left\langle -u\right| \otimes \left|\phi_{11}\right\rangle \left\langle 00\right| \\
&+ \tfrac{1}{\sqrt{2}} \left|+v\right\rangle \left\langle +v\right| \otimes \left|\phi_{10}\right\rangle \left\langle 00\right| \\
&+ \tfrac{1}{\sqrt{2}} i \left|-v\right\rangle \left\langle -v\right| \otimes \left|\phi_{01}\right\rangle \left\langle 00\right| . \qquad (14)
\end{aligned}
$$

We have now simplified enough to quickly complete the calculation:

$$
E_{ij} = \left(I_S \otimes \left\langle 0\right|_{M1} \otimes \left\langle 0\right|_{M2}\right) U^{\dagger} I_S \otimes \left|\phi_{ij}\right\rangle \left\langle \phi_{ij}\right| U \left(I_S \otimes \left|0\right\rangle_{M1} \otimes \left|0\right\rangle_{M2}\right) \qquad (15)
$$

$$
(16)
$$

and recognizing that each of the $|\phi_{ij}\rangle$ are orthogonal to the others (their projectors form a PVM), and that $\left(I_S \otimes \langle 0|_{M1} \otimes \langle 0|_{M2}\right) U^{\dagger}$ is just the conjugate of what we had for $U\left(I_S \otimes |0\rangle_{M1} \otimes |0\rangle_{M2}\right)$, we can insert everything and use the orthonormality of the $|\phi_{ij}\rangle$ to find that

$$
\begin{aligned}
E_{00} &= \frac{1}{2} \left|+u\right\rangle \left\langle +u\right| && (17) \\
E_{01} &= \frac{1}{2} \left|+v\right\rangle \left\langle +v\right| && (18) \\
E_{10} &= \frac{1}{2} \left|-v\right\rangle \left\langle -v\right| && (19) \\
E_{11} &= \frac{1}{2} \left|-u\right\rangle \left\langle -u\right| . && (20)
\end{aligned}
$$

Each of these are proportional to rank-1 projectors, with eigenvalues 1/2 and 0, so they are all positive operators. To see that they sum to the identity, calculate

$$
E_{00} + E_{01} + E_{10} + E_{11} = \frac{1}{2}\left(\left|+u\right\rangle \left\langle +u\right| + \left|-u\right\rangle \left\langle -u\right|\right) + \frac{1}{2}\left(\left|+v\right\rangle \left\langle +v\right| + \left|-v\right\rangle \left\langle -v\right|\right). \qquad (21)
$$

But $|+u\rangle \langle +u|$ and $|-u\rangle \langle -u|$ are each projectors onto orthogonal qubit states (they represent qubit states which are antipodal to each other on the Bloch sphere), so they must sum to the identity I, likewise for $|+v\rangle \langle +v|$ and $|-v\rangle \langle -v|$. So, we have that

$$
E_{00} + E_{01} + E_{10} + E_{11} = \frac{1}{2} I + \frac{1}{2} I = I. \qquad (22)
$$

Thus, the operators $\{E_{ij}\}_{ij}$ indeed form a POVM on S.

Okay, now we will consider the description of a new measurement based on the measurement just described above. The new measurement procedure is as as follows: perform the joint measurement procedure above, which returns outcome (i, j), and *ignore* the outcome j of measurement apparatus $M2$. Basic probability tells us that $\Pr(i) = \sum_j \Pr(i|j)$, where Pr(i|j) is the conditional probability of i, given that j is true. Our new POVM $\mathcal{F}$ will have elements $\{F_i\}$, $i \in \{0, 1\}$.

(c.) [2 Points] Write down an expression for $\Pr(i)$ for an arbitrary system state $|\chi\rangle_S$ as a single trace using the POVM elements E_{ij}. Deduce from this expressions for the operators F_i in terms of the operators E_{ij}.

Solution: Using the Born rule,

$$
\Pr(i) = \operatorname{Tr}\left(\left|\chi\right\rangle \left\langle \chi\right|_S F_i\right). \qquad (23)
$$

But we also know that $\Pr(i) = \sum_j \Pr(i|j)$, and that $\Pr(i|j) = \mathrm{Tr}\left(|\chi\rangle\langle\chi|_S E_{ij}\right)$. So, we see that

$$\Pr(i) = \mathrm{Tr}\left(|\chi\rangle\langle\chi|_S F_i\right) = \mathrm{Tr}\left(|\chi\rangle\langle\chi|_S E_{i0}\right) + \mathrm{Tr}\left(|\chi\rangle\langle\chi|_S E_{i1}\right) \tag{24}$$

$$\tag{25}$$

$$= \mathrm{Tr}\left(|\chi\rangle\langle\chi|_S (E_{i0} + E_{i1})\right). \tag{26}$$

So, we conclude that $F_i = E_{i0} + E_{i1}$, which obviously form a POVM because the Es do.

(d.) [2 Points] Show that we can also write $F_i = q|i\rangle\langle i| + (1-q)\frac{\mathbb{I}}{2}$ for some appropriate value of q (which you'll find in the process), and show that the operators $\{F_i\}$ form a POVM.

Solution: Using the Bloch sphere representation of the qubit density matrix, we know that

$$|+u\rangle\langle +u| = \frac{1}{2}(I + \frac{1}{\sqrt{2}}\sigma_x + \frac{1}{\sqrt{2}}\sigma_z) \tag{27}$$

$$|-u\rangle\langle -u| = \frac{1}{2}(I + \frac{1}{\sqrt{2}}\sigma_x + \frac{1}{\sqrt{2}}\sigma_z) \tag{28}$$

$$|+v\rangle\langle +v| = \frac{1}{2}(I - \frac{1}{\sqrt{2}}\sigma_x + \frac{1}{\sqrt{2}}\sigma_z) \tag{29}$$

$$|-v\rangle\langle -v| = \frac{1}{2}(I - \frac{1}{\sqrt{2}}\sigma_x + \frac{1}{\sqrt{2}}\sigma_z) \tag{30}$$

So, we see that

$$F_0 = \frac{1}{4}(I + \frac{1}{\sqrt{2}}\sigma_x + \frac{1}{\sqrt{2}}\sigma_z) + \frac{1}{4}(I - \frac{1}{\sqrt{2}}\sigma_x + \frac{1}{\sqrt{2}}\sigma_z) \tag{31}$$

$$= \frac{1}{4}(2I + \sqrt{2}\sigma_z) \tag{32}$$

$$F_1 = \frac{1}{4}(I - \frac{1}{\sqrt{2}}\sigma_x - \frac{1}{\sqrt{2}}\sigma_z) + \frac{1}{4}(I + \frac{1}{\sqrt{2}}\sigma_x - \frac{1}{\sqrt{2}}\sigma_z) \tag{33}$$

$$= \frac{1}{4}(2I - \sqrt{2}\sigma_z) \tag{34}$$

Resolving σ_z into its spectral decomposition, $\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$, we can write

$$F_0 = \frac{1}{4}(2I + \sqrt{2}(|0\rangle\langle 0| - |1\rangle\langle 1|) - 2\sqrt{2}\,|0\rangle\langle 0| + 2\sqrt{2}\,|0\rangle\langle 0|) \tag{35}$$

$$= \frac{1}{4}(2I - \sqrt{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) + 2\sqrt{2}\,|0\rangle\langle 0|) \tag{36}$$

$$= \frac{1}{4}(2I - \sqrt{2}I + 2\sqrt{2}\,|0\rangle\langle 0|) \tag{37}$$

$$= \frac{1}{\sqrt{2}}|0\rangle\langle 0| + (1 - \frac{1}{\sqrt{2}})\frac{I}{2} \tag{38}$$

Similarly, we find that

$$F_1 = \frac{1}{\sqrt{2}}|1\rangle\langle 1| + (1 - \frac{1}{\sqrt{2}})\frac{I}{2}. \tag{39}$$

As mentioned at the end of (c.), it's clear that these form a POVM. We can nonetheless verify explicitly here by noting that each F_i is positive because each is a sum of positive operators with positive coefficients, and adding the two obviously gives the identity operator as required.

(e.) [1 Point] Give an *interpretation* for the measurement procedure that POVM $\{F_i\}$ represents that allows us to understand it as a *noisy* measurement of σ_z. This interpretation should be something like 'with probability p, a measurement of *insert measurement here* is performed, and with probability $1-p$, the POVM *insert POVM here* is measured'. (Hint: the POVM should be able to be interpreted as just ignoring the system and flipping a fair coin.)

Solution: This POVM is a probabilistic mixture of the *PVM* $\{P_0 = |0\rangle\langle 0|, P_1 = |1\rangle\langle 1|\}$ which measures σ_z, and the POVM $\{V_0 = \frac{I}{2}, V_1 = \frac{I}{2}\}$ which does absolutely nothing to the system, and gives each outcome with 50 percent likelihood. How do we see this? Well, first, let's think about the POVM $\{V_0 = \frac{I}{2}, V_1 = \frac{I}{2}\}$. For any state ρ, the outcomes are 50/50 because $\mathrm{Tr}\,(\rho\frac{I}{2}) = \frac{1}{2}$, and resolving each $V_i = \frac{I}{2}$ into measurement operators $M_i = \frac{I}{\sqrt{2}}$ such that $V_i = M_i^\dagger M_i$, the measurement update rule tells us the post-measurement state should be

$$\frac{M_i \rho M_i^\dagger}{\mathrm{Tr}\,(\rho\frac{I}{2})} = \frac{I\rho I}{2\frac{1}{2}} = \rho, \tag{40}$$

the state is completely unchanged, regardless of what the measurement outcome was! So if instead of actually measuring your state ρ, you just flipped a fair coin and used that as your 'measurement outcome', the POVM $\{V_0 = \frac{I}{2}, V_1 = \frac{I}{2}\}$ accurately describes this operation.

Now, suppose that we had some machine that we put our quantum state ρ that we want to measure in, and we push a button, it measures ρ, and tells us outcome 0 or outcome 1. We want it to measure σ_z, but what actually goes on inside the machine is that with probability $\frac{1}{\sqrt{2}}$ it measures σ_z as a PVM and tells us the outcome (0 or 1), and with probability $1 - \frac{1}{\sqrt{2}}$ it does absolutely nothing to the state, flips a fair coin, and tells us it was outcome 0 if it was heads and outcome 1 if it was tails. What should the outcome probabilities be (for either 0 or 1)? We can take a probabilistically weighted sum of Born rules:

$$\begin{aligned}
\Pr(0) &= \frac{1}{\sqrt{2}}\mathrm{Tr}\,(\rho\,|0\rangle\langle 0|) + (1 - \frac{1}{\sqrt{2}})\mathrm{Tr}\,(\rho\frac{I}{2}) && (41)\\
&= \mathrm{Tr}\,\left(\rho\left(\frac{1}{\sqrt{2}}|0\rangle\langle 0| + \left(1 - \frac{1}{\sqrt{2}}\right)\frac{I}{2}\right)\right) && (42)\\
&= \mathrm{Tr}\,(\rho F_0) && (43)\\
\Pr(1) &= \frac{1}{\sqrt{2}}\mathrm{Tr}\,(\rho\,|1\rangle\langle 1|) + (1 - \frac{1}{\sqrt{2}})\mathrm{Tr}\,(\rho\frac{I}{2}) && (44)\\
&= \mathrm{Tr}\,\left(\rho\left(\frac{1}{\sqrt{2}}|1\rangle\langle 1| + \left(1 - \frac{1}{\sqrt{2}}\right)\frac{I}{2}\right)\right) && (45)\\
&= \mathrm{Tr}\,(\rho F_1) && (46)
\end{aligned}$$

and so we see that F_0 and F_1, as a POVM, reproduce the measurement statistics of this interpreted imperfect measurement of σ_z. It is unnecessary, but one could also go through the exercise of decomposing F_0 and F_1 into their measurement/Kraus operators and calculate the updated state ρ, and would see that the updated state is always a mixed state that is with probability $\frac{1}{\sqrt{2}}$ the state ρ projected onto $|i\rangle$ (which would be what happened if σ_z were actually measured and outcome i attained), and with probability $1 - \frac{1}{\sqrt{2}}$ is left as ρ (as it should be if the machine did nothing)!

(f.) [1 Point] Now, go back to the definition of $\mathcal{F}$ given before part (c.), and instead consider the case that when we get measurement outcome (i, j) we instead ignore the outcome i of measurement apparatus $M1$ and keep j, calling this POVM $\mathcal{G} = \{G_j\}$. You need not show all of the work for parts (c.), (d.), and (e.) again, but give an analogous interpretation to that given in part (e.) but for $\mathcal{G}$. (It should just be a difference in the observable measured with probability p.)

Solutions: I won't repeat the work, as it is almost identical to that above, but in this case one would find POVM elements $G_j = E_{0j} + E_{1j}$, which could then be rewritten as $G_0 = \frac{1}{\sqrt{2}} |+\rangle\langle+| + \left(1 - \frac{1}{\sqrt{2}}\right) \frac{I}{2}$ and $G_0 = \frac{1}{\sqrt{2}} |-\rangle\langle-| + \left(1 - \frac{1}{\sqrt{2}}\right) \frac{I}{2}$, and the interpretation would be that with probability $\frac{1}{\sqrt{2}}$ the PVM $\{|+\rangle\langle+|, |-\rangle\langle-|\}$ which measures σ_x is performed, and with complementary probability, nothing is done and a coin is flipped.

One can now conclude that the POVM $\{E_{ij}\}$ is a *noisy joint measurement* of σ_x and σ_z on the *single qubit system* S that gives *incomplete information* about about the two incompatible observables.

2 Kraus [5 Points]

Quantum information theory uses the CNOT or controlled-not gate, a unitary operation on two qubits that acts as

$$\begin{aligned} U_{\text{cnot}}: \quad & |00\rangle \to |00\rangle & (47)\\ & |01\rangle \to |01\rangle & (48)\\ & |10\rangle \to |11\rangle & (49)\\ & |11\rangle \to |10\rangle . & (50) \end{aligned}$$

In other words, the first ("control") qubit stays fixed, while the second ("target") qubit flips when the first is a 1, and stays fixed when the first is a 0.

(a.) [1 Point] Find the Kraus operators for the quantum channel $\mathcal{E}_{CNOT}$, where the system qubit is the control qubit, and the environment is the target qubit. Assume that the target/environment qubit starts in state $|0\rangle\langle 0|$

Solution: Let the 'control qubit' of the CNOT operation (which our channel will act on) be described by the state $\rho_C = |\psi\rangle\langle\psi| = |\alpha|^2 |0\rangle\langle 0| + \alpha\beta^* |0\rangle\langle 1| + \alpha^*\beta |1\rangle\langle 0| + |\beta|^2 |1\rangle\langle 1|$, where $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. Then the initial state of the whole system is $\rho = \rho_C \otimes |0\rangle\langle 0|$. We can evolve the entire system unitarily with the unitary U_{cnot}:

$$\begin{aligned} U_{\text{cnot}}\rho_C \otimes |0\rangle\langle 0| U_{\text{cnot}}^\dagger &= U_{\text{cnot}} \left(|\alpha|^2 |00\rangle\langle 00| + \alpha\beta^* |00\rangle\langle 10| + \alpha^*\beta |10\rangle\langle 00| + |\beta|^2 |10\rangle\langle 10|\right) U_{\text{cnot}}^\dagger \\ &= |\alpha|^2 |00\rangle\langle 00| + \alpha\beta^* |00\rangle\langle 11| + \alpha^*\beta |11\rangle\langle 00| + |\beta|^2 |11\rangle\langle 11| . \qquad (51) \end{aligned}$$

We can now take the partial trace over the target/environment qubit to obtain $\mathcal{E}(\rho_C)$:

$$\begin{aligned} \mathcal{E}(\rho_C) &= \text{Tr}_E \left(U_{\text{cnot}}\rho_C \otimes |0\rangle\langle 0| U_{\text{cnot}}^\dagger\right) & (52)\\ &= |\alpha|^2 |0\rangle\langle 0| + |\beta|^2 |1\rangle\langle 1| . & (53) \end{aligned}$$

Comparing this to the initial density matrix of our qubit, $\rho_C = |\psi\rangle\langle\psi| = |\alpha|^2 |0\rangle\langle 0| + \alpha\beta^* |0\rangle\langle 1| + \alpha^*\beta |1\rangle\langle 0| + |\beta|^2 |1\rangle\langle 1|$, we see that this is the same as

$$|0\rangle\langle 0| \rho |0\rangle\langle 0| + |1\rangle\langle 1| \rho |1\rangle\langle 1| . \tag{54}$$

Hence, we can see that Kraus operators $K_0 = |0\rangle\langle 0|$ and $K_1 = |1\rangle\langle 1|$ can be used to implement the quantum channel $\mathcal{E}$.

One important example of a quantum channel is the depolarizing channel. In general, the depolarizing channel on a qudit with Hilbert space $\mathbb{C}^d$ can be written

$$\mathcal{E}(\rho) = (1-p)\rho + p\frac{\mathbb{I}}{d}, \tag{55}$$

giving the interpretation that with probability p, the input state is replaced by a completely unknown random state (which we must represent with the maximally mixed state $\frac{\mathbb{I}}{d}$), and otherwise does nothing.

(b.) [2 Points] Consider the depolarizing channel for a qubit ($d = 2$). Find a set of Kraus operators for this channel. (Hint: Consider the decomposition of the qubit density operator into Pauli operators via the Bloch sphere. What does conjugating a Pauli operator by another Pauli operator do? Using this information, you should be able to write the identity operator as a linear combination of an arbitrary density operator, plus conjugations by Paulis.)

Solution: Start with the Bloch sphere decomposition of the qubit density matrix described by Bloch vector $\vec{v}$, $\rho = \frac{1}{2}(I + v_x\sigma_x + v_y\sigma_y + v_z\sigma_z)$. Suppose we conjugate by any of the Paulis σ_i, then we obtain

$$\sigma_x\rho\sigma_x = \frac{1}{2}(I + v_x\sigma_x - v_y\sigma_y - v_z\sigma_z) \tag{56}$$

$$\sigma_y\rho\sigma_y = \frac{1}{2}(I - v_x\sigma_x + v_y\sigma_y - v_z\sigma_z) \tag{57}$$

$$\sigma_z\rho\sigma_z = \frac{1}{2}(I - v_x\sigma_x - v_y\sigma_y + v_z\sigma_z) \tag{58}$$

which are calculated using the observation that $\sigma_i\sigma_j\sigma_i = -\sigma_j$ for $i \neq j$. Taking an equally weighted linear combination of all three yields

$$\sigma_x\rho\sigma_x + \sigma_y\rho\sigma_y + \sigma_z\rho\sigma_z = 2I - \frac{1}{2}(I + v_x\sigma_x + v_y\sigma_y + v_z\sigma_z) \tag{59}$$

$$= 2I - \rho \tag{60}$$

So, we have then that

$$\frac{I}{2} = \frac{1}{4}(\rho + \sigma_x\rho\sigma_x + \sigma_y\rho\sigma_y + \sigma_z\rho\sigma_z) \tag{61}$$

Plugging this identity into the equation for the depolarizing channel, we get

$$\mathcal{E}(\rho) = (1-p)\rho + p\frac{1}{4}(\rho + \sigma_x\rho\sigma_x + \sigma_y\rho\sigma_y + \sigma_z\rho\sigma_z) \tag{62}$$

$$= \sqrt{1-\frac{3}{4}p}\rho\sqrt{1-\frac{3}{4}p} + \frac{\sqrt{p}\sigma_x}{2}\rho\frac{\sqrt{p}\sigma_x}{2} + \frac{\sqrt{p}\sigma_y}{2}\rho\frac{\sqrt{p}\sigma_y}{2} + \frac{\sqrt{p}\sigma_z}{2}\rho\frac{\sqrt{p}\sigma_z}{2}, \tag{63}$$

so we can see that an appropriate set of Kraus operators are $\{K_0 = \sqrt{1-\frac{3}{4}p}I, K_1 = \frac{\sqrt{p}}{2}\sigma_x, K_2 = \frac{\sqrt{p}}{2}\sigma_y, K_3 = \frac{\sqrt{p}}{2}\sigma_z\}$

(c.) [1 Point] By studying the action of the depolarizing channel on the Bloch sphere representation of a qubit, describe what this channel does to the Bloch sphere.

Solution: We have that

$$\mathcal{E}(\rho) = (1-p)\rho + p\frac{I}{2} \tag{64}$$

$$= (1-p)\frac{I + v_x\sigma_x + v_y\sigma_y + v_z\sigma_z}{2} + p\frac{I}{2} \tag{65}$$

$$= \frac{I + (1-p)v_x\sigma_x + (1-p)v_y\sigma_y + (1-p)v_z\sigma_z}{2} \tag{66}$$

So, we see that the depolarizing channel takes a qubit state with Bloch vector $\vec{v} = (v_x, v_y, v_z)$ with magnitude $\|\vec{v}\| = \sqrt{v_x^2 + v_y^2 + v_z^2}$ to a qubit state with Bloch vector $\vec{v}' = ((1-p)v_x, (1-p)v_y, (1-p)v_z) = (1-p)\vec{v}$, whose magnitude is $\|\vec{v}'\| = \sqrt{1-p}\|\vec{v}\| < \|\vec{v}\|$. So, applying the depolarizing channel shrinks the Bloch vector of the qubit state until its Bloch vector is zero, in which case the state is the maximally mixed state $\rho = \frac{I}{2}$.

(d.) [1 Point] While the interpretation given makes it seem like p should be between 0 and 1, this actually need not be true for the formula given for the depolarizing channel to correctly represent a quantum channel. Find the most general bounds on p which allow for $\mathcal{E}(\rho) = (1-p)\rho + p\frac{\mathbb{I}}{d}$ to be a quantum channel for $d = 2$.

Solution: Looking at the Kraus operators $\{K_0 = \sqrt{1 - \frac{3}{4}p}I, K_1 = \frac{\sqrt{p}}{2}\sigma_x, K_2 = \frac{\sqrt{p}}{2}\sigma_y, K_3 = \frac{\sqrt{p}}{2}\sigma_z\}$, we see that if $p < 0$ then the last three are no longer positive, which they must be. So, $p \geq 0$. But the first Kraus operator retains positivity for p up to 4/3, after which it is no longer positive, so we see that we must have $p \in [0, 4/3]$.

3 Stinespring and Kraus [5 points]

It was discussed in class that the unitary evolution of a density matrix is not the most general way a quantum system can evolve, but rather a *quantum channel* represented by the CPTP map $\mathcal{E} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ is. We explored the Kraus representation of such a quantum channel and saw that this kind of map bears a similarity to a mixture of unitary evolutions (that is, a density matrix ρ can in general evolve to become a mixture of density matrices under a quantum channel). It was demonstrated in class that in appropriate situations, Kraus operators describing the evolution of a subsystem can be obtained by considering unitary evolution of a larger system, and then tracing out everything except the subsystem of interest (via the partial trace).

Kraus operators do not always need to be obtained in this fashion, but there is a theorem from operator theory called the Stinespring Dilation Theorem that says you always can if you want to. In our language, the Stinespring Dilation Theorem says: *Any CPTP map $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A)$ can be expressed as the reduced action of a single unitary operator acting on an extended Hilbert space $U : \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_A \otimes \mathcal{H}_B$.* That is, we can write

$$\mathcal{E}(\rho) = \mathrm{Tr}_B(U\rho \otimes \sigma U^\dagger), \tag{67}$$

where σ is some state on an environment Hilbert space $\mathcal{H}_B$ of unspecified dimension.

Show that this is true for a quantum channel $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A)$ on a d-dimensional quantum system $\mathcal{H}_A = \mathbb{C}^d$, by explicitly constructing the unitary U in terms of the Kraus operators $\{K_i\}_{i=0}^{k-1}$ of $\mathcal{E}$, assuming initial environment state $\sigma = |0\rangle\langle 0|$. (Hint: Similarly to how there is a unitary freedom in choosing Kraus operators to implement a quantum channel, the unitary matrix U here is not unique. In fact, you should be able to argue that only d columns of U are fixed by the Kraus operators. If you choose to reverse the ordering of your Hilbert spaces (and hence their bases) to $\mathcal{H}_B \otimes \mathcal{H}_A$ instead of $\mathcal{H}_A \otimes \mathcal{H}_B$, it will be exactly the first d columns of $U : \mathcal{H}_B \otimes \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_A$ that will be fixed by your Kraus operators, and that part of your unitary will have a nice block form. You can choose to work in this reversed basis for convenience if you wish. The remaining columns of U can be chosen arbitrarily as long as all of the columns end up orthonormal).

Solution: For convenience, let's work with the tensor factor ordering $\mathcal{H}_B \otimes \mathcal{H}_A$. If $\dim(\mathcal{H}_B) = b$, then we can take orthonormal basis $\{|i\rangle_B\}_{i=0}^{b-1}$ for $\mathcal{H}_B$ and orthonormal basis $\{|i\rangle_A\}_{i=0}^{d-1}$ for $\mathcal{H}_A$, so that we have orthonormal basis $\{|i\rangle_B |j\rangle_A\}_{i,j}^{b-1,d-1}$. Note that the convention for ordering of such a tensor basis is such that i and j start at 0 and the index j indexing $\mathcal{H}_A$'s basis increases until it gets to $d-1$, then i indexing $\mathcal{H}_B$'s basis increases by 1, j rolls back to zero, and we iterate through j again. That is, the ordered basis can be written $\{|0\rangle_B |0\rangle_A, |0\rangle_B |1\rangle_A, ..., |0\rangle_B |d-1\rangle_A, |1\rangle_B |0\rangle_A, |1\rangle_B |1\rangle_A, ..., |b-1\rangle_B |d-1\rangle_A\}$. The ordering of this basis determines the positions of rows/columns of matrices representing operators that act on $\mathcal{H}_B \otimes \mathcal{H}_A$.

Now, we're given a set of Kraus operators (let's say that there's k of them) $\{K_i\}_{i=0}^{k-1}$ which act on $\mathcal{H}_A$, and implement a quantum channel on $\mathcal{H}_A$. That is, for state $\rho \in \mathcal{L}(\mathcal{H}_A)$, the action of the channel can be written $\mathcal{E}(\rho) = \sum_{i=0}^{k-1} K_i \rho K_i^\dagger$. Because channels preserve trace (as they must in order to map quantum states to quantum states again), we know that the Kraus operators satisfy the relation $\sum_{i=0}^{k-1} K_i^\dagger K_i = I$. Let's start by asking: is there a matrix $U \in \mathcal{L}(\mathcal{H}_B \otimes \mathcal{H}_A)$ that we could choose such that

$$\mathrm{Tr}_B \left(U |0\rangle\langle 0|_B \otimes \rho U^\dagger \right) = \mathcal{E}(\rho) = \sum_{i=0}^{k-1} K_i \rho K_i^\dagger ? \tag{68}$$

Taking the partial trace, we find that

$$\mathrm{Tr}_B \left(U |0\rangle\langle 0|_B \otimes \rho U^\dagger \right) = \sum_{r=0}^{b-1} (\langle r|_B \otimes I_A) U (|0\rangle\langle 0|_B \otimes \rho) U^\dagger (|r\rangle_B \otimes I_A) \tag{69}$$

$$= \sum_{r=0}^{b-1} (\langle r|_B \otimes I_A) U (|0\rangle_B \otimes I_A) \rho (\langle 0|_B \otimes I_A) U^\dagger (|r\rangle_B \otimes I_A). \tag{70}$$

This suggests that if we took $b = k$, maybe choosing U such that $K_r = (\langle r|_B \otimes I_A) U (|0\rangle_B \otimes I_A)$ could work. Let's work this expression out explicitly to see what the relationship between matrix elements of U and matrix elements of each K_r would have to be. To do this, we can write U in terms of the standard basis for $\mathcal{L}(\mathcal{H}_B \otimes \mathcal{H}_A)$,

$$U = \sum_{i,j,k,l} U_{ijkl} |ij\rangle_{BA} \langle kl|_{BA} = \sum_{i,j,k,l} U_{ijkl} |i\rangle\langle k|_B \otimes |j\rangle\langle l|_A \tag{71}$$

where U_{ijkl} is the matrix element at the ijth row and klth column (i and k range from 0 to

$b-1$, where j and l range from 0 to $d-1$). Let's plug this into our expression:

$$(\langle i|_B \otimes I_A)U(|0\rangle_B \otimes I_A) = \sum_{i,j,k,l} U_{ijkl}(\langle r|_B \otimes I_A) |i\rangle \langle k|_B \otimes |j\rangle \langle l|_A (|0\rangle_B \otimes I_A) \tag{72}$$

$$= \sum_{i,j,k,l} U_{ijkl}\langle r|i\rangle\langle k|0\rangle |j\rangle \langle l|_A \tag{73}$$

$$= \sum_{i,j,k,l} U_{ijkl}\delta_{r,i}\delta_{k,0} |j\rangle \langle l|_A \tag{74}$$

$$= \sum_{j,l=0}^{d-1} U_{rj0l} |j\rangle \langle l|_A . \tag{75}$$

Then, expanding the matrix K_r in the standard basis as $K_r = \sum_{j,l=0}^{d-1} K_{r_{jl}} |j\rangle \langle l|_A$, equating $K_r = (\langle i|_B \otimes I_A)U(|0\rangle_B \otimes I_A)$ yields

$$\sum_{j,l=0}^{d-1} K_{r_{jl}} |j\rangle \langle l|_A = \sum_{j,l=0}^{d-1} U_{rj0l} |j\rangle \langle l|_A \tag{76}$$

which is true if and only if $K_{r_{jl}} = U_{rj0l}$. So we see that by choosing matrix element U_{rj0l} of U to be $K_{r_{jl}}$, we will have that

$$\mathrm{Tr}_B \left(U |0\rangle \langle 0|_B \otimes \rho U^\dagger\right) = \sum_{i=0}^{k-1} K_i \rho K_i^\dagger = \mathcal{E}(\rho) \tag{77}$$

as desired. We can see, then, that the $0l$th column of U is the lth column of each Kraus operator stacked on top of each other, and iterating l from 0 to $d-1$ we see that the Kraus operators fix exactly the first d columns of the matrix U, and it doesn't matter what any of the remaining unfixed matrix elements of U are, the relation $\mathrm{Tr}_B \left(U |0\rangle \langle 0|_B \otimes \rho U^\dagger\right) = \mathcal{E}(\rho)$ will hold. But, what about U being unitary? The whole point is that we want to say that what appears to be a non-unitary evolution on quantum system $\mathcal{H}_A$ is actually a unitary evolution on the larger system $\mathcal{H}_B \otimes \mathcal{H}_A$, so that the axiom of quantum mechanics that all time evolution is unitary is preserved. For U to be unitary, we need for the inner products of the columns of U are orthonormal to each other. Is this true for the d columns that are fixed by the Kraus operators? The (complex) inner product of the mth column of U with the nth column is

$$\sum_{r,j=0}^{d-1} U^*_{rj0m} U_{rj0n} = \sum_{r=0}^{d-1} \sum_{j=0}^{d-1} K^*_{r_{jm}} K_{r_{jn}} . \tag{78}$$

Now, observe that $\sum_{j=0}^{d-1} K^*_{r_{jm}} K_{r_{jn}}$ is the mn entry of the matrix $K_r^\dagger K_r$. So we see that the inner product of the m and nth columns of U is equal to the mn entry of the matrix $\sum_{r=0}^{d-1} K_r^\dagger K_r$. But we know that because K_r are Kraus operators for a quantum channel that $\sum_{r=0}^{d-1} K_r^\dagger K_r = I_A$, so we see that the inner product of the m and nth columns of U are equal to 1 if $n = m$, and 0 otherwise. So, the columns are orthonormal, as they would need to be for U to be unitary. The remaining $db-d$ columns of U can be anything we want, so in particular, we can use the Gram-Schmidt orthogonalization prodecure to come up with $db-d$ columns to put in the unspecified part of the unitary that are orthonormal to themselves as well as the first d columns, and U will be unitary. Thus we have shown that for any quantum channel, there exists a unitary U acting on a larger Hilbert space that implements this quantum channel.
