Ph125c Spring 2017
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Homework - 4
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Due: 5:00pm, 5/10/2017

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1 The Mach-Zehnder Interferometer [9 Points]

In the discussion of interference in textbooks, one typically considers the nature of the interference pattern when one does or does not learn about the path information. In this problem, we will consider a scenario wherein one can acquire partial path information. We will consider two cases: one wherein the experimenter acquires the path information, and the other wherein this information is encoded in the environment, but is not available to the experimenter.
A stripped-down version of the double-slit experiment is the Mach-Zehnder interferometer, shown in Figure 1.

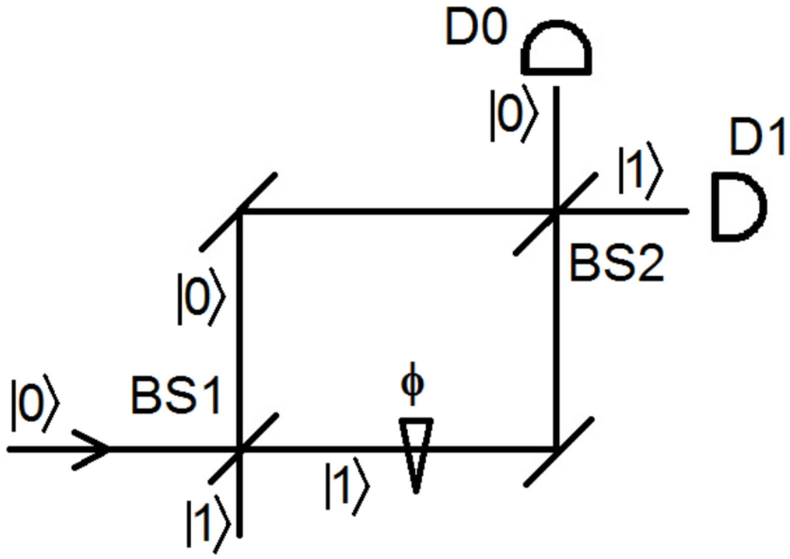

Figure 1: A Mach-Zehnder Interferometer

Photons are made to enter the interferometer one at a time, so there is at most one photon in the apparatus at any time; the path degree of freedom forms a two-level system labelled by $|0\rangle, |1\rangle$. We will call this the 'which-path' basis. $|0\rangle$ indicates that the photon is travelling in the upper arm of the interferometer, whereas $|1\rangle$ indicates that it is in the lower arm. The first beamsplitter, BS1, splits the input path $|0\rangle$ into a coherent superposition of two paths within the interferometer. A phase shifter placed in one of these paths allows the experimenter to change the relative phase $e^{i\phi}$ between $|0\rangle$ and $|1\rangle$ introduced between the arms of the interferometer, which are then recombined at a second beamsplitter BS2. Photons are detected by photodetectors D0, D1 in the output arms. If the input and output arms are labelled as in the figure, the unitary U_p describing each of the two (identical) beamsplitters and the unitary U_p describing the phase shifter are given by

$$U_b = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\langle 0| + \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\langle 1| \tag{1}$$

$$U_p = |0\rangle\langle 0| + e^{i\phi}|1\rangle\langle 1|. \tag{2}$$

The photodetectors make a measurement of how the photon exited the interferometer, and they measurement they perform are described by the PVM $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$. You should be able to convince yourself that for input in just one arm, corresponding to the input state $|0\rangle$, the probabilities of detection at D0, D1 respectively are given by

$$P(D0) = \frac{1}{2}(1 + \cos\phi) \tag{3}$$

$$P(D1) = \frac{1}{2}(1 - \cos\phi). \tag{4}$$

In an ideal interference experiment, the internal paths recombine coherently at BS2, and by choosing the phase appropriately, the experimenter can vary from total destructive to total constructive interference in a given output arm. This is what we see in the above equations. Note that the combination of the second beamsplitter plus detectors yields an effective measurement on the photon in the region between the beamsplitters (say, just before it reaches BS2) in a basis $|+\rangle = \{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\}$. That is, we can lump BS2 and the photodetectors into a single device, which measures the PVM $\{|+\rangle\langle +|, |-\rangle\langle -|\}$ inside the interferometer (after the photon has gone through the phase shifter, if it took the lower path). This basis is complementary to the basis containing path information. Decoherence in the 'which-path' ($|0\rangle$ and $|1\rangle$) basis has observable effects in the measurement statistics in this complementary basis. If the contributions from different internal paths become partially or fully incoherent, the visibility of the interference fringes (which is seen by doing repeated experiments and comparing the frequency of D0 clicks vs D1 clicks) is reduced, or the interference disappears altogether.

Now suppose we introduce a probe system that 'learns' partial path information in the following way. The probe interacts with the photon if it travels down path 1 (after the phase shifter). We treat the probe also as a quantum system, initially in a pure state $|P0\rangle$. As a result of the interaction between the photon and the probe, by the time the photon reaches BS2, the probe either remains in state $|P0\rangle$ (if the photon was going down path 0) or has evolved into state $|P1\rangle$ (if the photon was going down path 1). We can describe this coupling via an interaction unitary acting after BS1 and the phase shifter as follows:

$$U = (|0\rangle\langle 0|)_S \otimes I_P + (|1\rangle\langle 1|)_S \otimes (|P_1\rangle\langle P_0| + |P_1^{\perp}\rangle\langle P_0|^{\perp})_P \tag{5}$$

where $|P_0^{\perp}\rangle$, $|P_1^{\perp}\rangle$ are states orthogonal to $|P0\rangle$, $|P1\rangle$ respectively, and we use the labels S, P to refer to the photon path system and probe, respectively, when it is not clear from the context to which system we are referring.

(a.) [2 Points] We first consider the case where the experimenter does not measure the environment. The interaction with the environment, even if the experimenter does not access it, causes a change in the measurement probabilities and the visibility of interference that one would see by changing ϕ. Assume input state $|0\rangle_S$ to the interferometer, as before. What is the state (i.e. the reduced density operator) describing the photon path degree of freedom before BS2? Write the state in the $|0\rangle_S$, $|1\rangle_S$ basis, as a function of the overlap $\langle P_0|P_1\rangle$ of the probe states; note the possibility for reduced coherence in this basis (the reduction of the off-diagonal elements of the density matrix) due – as we will argue below – to the partial path information now contained in the environment.

(b.) [1 Point] Find the probability of getting a click in detector D0, as a function of ϕ and the overlap $\langle P_0|P_1\rangle$. This causes visibility of interference to be reduced (as seen in the next problem).

(c.) [3 Points] Above, we were considering the case where the experimenter directly measures the internal system of the interferometer (where the photon is in some quantum superposition of travelling down either path), with the PVM $\{|+\rangle\langle+|, |-\rangle\langle-|\}$ in order to observe interference between the two paths which can be modulated by changing ϕ. Now consider the case in which the experimenter measures the probe system while the photon is in the interferometer (after interacting with the probe, but before reaching BS2) in order to try and infer what path of the interferometer the photon is in (the goal of this measurement is NOT to observe interference, as in the previous part). Since we are not concerned with interference in this part, and all measurements are occuring before the photon hits BS2, you can pretend that BS2 and the photodetectors do not exist in this part of the problem. It is useful to introduce an orthonormal basis $|0\rangle_P$, $|1\rangle_P$ for the probe system, such that the probe states may be written

$$|P_0\rangle = \cos\theta\,|0\rangle_P + \sin\theta\,|1\rangle_P \tag{6}$$

$$|P_1\rangle = \cos\theta\,|0\rangle_P - \sin\theta\,|1\rangle_P \tag{7}$$

where 2θ is the real angle parameterizing the overlap between the probe states: $\langle P_0|P_1\rangle = \cos(2\theta)$. It may be shown (although you are not being asked to) that the measurement which optimally discriminates between the non-orthogonal probe states, in the sense of minimizing the error in guessing the state, is then a measurement in the $|\pm\rangle_P = \frac{1}{\sqrt{2}}(|0\rangle_P \pm |1\rangle_P)$ basis. If the experimenter does measure the probe with this optimal measurement, he acquires information about the which-way observable (which branch of the interferometer the photon is in). How much information is gained may be quantified in terms of his probability of successfully predicting which path the photon is actually in, based on his probe measurement results. The experimenter's measurement on the probe system is described by the PVM $\{Q_0 = |+\rangle\langle+|_P, Q_1 = |-\rangle\langle-|_P\}$, where outcome i is taken to imply the corresponding outcome i in a which-way measurement. Show that the experimentalist's total probability of success in inferring the path the photon took based on the probe measurement readout is $\frac{1}{2}(1+\sin(2\theta))$. This shows that as the probe states become closer to orthogonal, the experimentalist can become near certain of which path the photon takes in the interferometer (and thus acquiring more information about this observable via the probe).

(d.) [3 Points] The combination of interaction with the probe, followed by measurement of the probe, as we have seen previously, can be described as a measurement on the photon path system alone (much in the same way as was done in problem 1a of homework 3). Find the 1-parameter family of POVMs (parameterized by θ) describing the effective measurement on the system. Comment on the limits where $|P_0\rangle$ and $|P_1\rangle$ are (i.) co-linear, (ii.) orthogonal.

Here's the punchline: in part (b.), you see that the mere presence of the probe, which isn't even being measured but the photon interacts with, changes the experimenter's ability to observe interference between the paths of the interferometer (which he would do by measuring $\{|+\rangle\langle+|, |-\rangle\langle-|\}$ on the photon). As the probe states become closer to orthogonal, the ability of changing ϕ to change the relative measurement probabilities decreases, hence the experimenter's ability to cause interference between the two different path possibilities in the interferometer is being reduced. In part (c.), we have seen that as the probe states $|P_0\rangle$, $|P_1\rangle$ become more distinguishable (closer to orthogonal), the environment contains more information about the which-way observable (in the sense that if the experimenter chooses to access to it by measurement, they can predict outcomes of measurement of this observable correctly with higher probability). So, we conclude that as the environment learns more information about some basis of the system (even if the experimenter does not access it), the evolution of the system changes from unitary, coherent evolution on the one hand, to decoherent evolution on the other.

2 The Quantum Eraser [11 Points]

The decoherence that a system experiences by virtue of its interaction with the environment is often considered to be the explanation of the emergence of classicality for the system. For instance, it is decoherence that ensures that orthogonal states of a pointer on an apparatus do not interfere. Such decoherence is important for understanding how the collapse postulate for measurements can be recovered when the apparatus is treated quantum mechanically, as is done in the de Broglie-Bohm interpretation, dynamical collapse models and the Everett interpretation (some of which you may hear about in the next few weeks).

In the previous problem, we saw that the amount of coherence a system exhibits relative to some basis decreases as one increases the amount of information the environment has about that basis. In this question, you will show that by measuring different observables on the environment, one can obtain information about different observables on the system. In particular, we will reconsider the Mach-Zehnder interferometer experiment discussed in the previous problem. We will see that when the photon and the probe become entangled, the probe can simultaneously carry information about the which-way observable and the complementary observable (relative phase) of the photon, and that interference is recovered if one chooses to measure the latter. This is known as the 'quantum eraser' experiment.

We will also consider the case wherein experimenters only have access to a small part of a large environment and we will see that in this case, the loss of coherence for the system can become effectively irreversible. In other words, we will express the condition for irreversibility in terms of how information about the system is encoded in the environment.

Recall The Mach-Zehnder interferometer that you became acquainted with in problem 1. After BS1 and the phase-shifter, the photon is in state $\frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi}|1\rangle)$. A probe that starts out in the state $|P_0\rangle$ is, again, coupled to the photon via the unitary

$$U = (|0\rangle\langle 0|)_S \otimes I_P + (|1\rangle\langle 1|)_S \otimes (|P_1\rangle\langle P_0| + |P_1^\perp\rangle\langle P_0|^\perp)_P \tag{8}$$

such that the state of the photon and probe after the interaction (but prior to BS2) is

$$|\Psi\rangle_{SP} = \frac{1}{\sqrt{2}}(|0\rangle_S|P_0\rangle_P + e^{i\phi}|1\rangle_S|P_1\rangle_P). \tag{9}$$

We will again assume that the probe states have the same form as in problem 1 (c.) in terms of the real parameter θ. We presume that after this interaction, the photon passes through BS2 and we register which detector fires. The experiment is repeated a large number of times, such that we obtain the relative frequency of the two detectors firing. As saw in the previous problem, the probability for each outcome, as a function of the relative phase ϕ between the arms of the intereferometer is

$$P(D0) = \frac{1}{2}(1 + \text{Re}(e^{i\phi}\langle P_0|P_1\rangle)) = \frac{1}{2}(1 + \langle P_0|P_1\rangle \cos\phi) \tag{10}$$

$$P(D1) = \frac{1}{2}(1 - \text{Re}(e^{i\phi}\langle P_0|P_1\rangle)) = \frac{1}{2}(1 - \langle P_0|P_1\rangle \cos\phi). \tag{11}$$

We define the fringe visibility V as

$$V = \frac{P_{max} - P_{min}}{P_{max} + P_{min}}, \tag{12}$$

where $P_{max} = \max_\phi P(D0)$, and similarly for P_{min}. We see that the fringe visibility in the presence of the probe system is given by

$$V = \langle P_0|P_1\rangle. \tag{13}$$

We have the maximum value of fringe visibility (perfect interference) when $\langle P_0|P_1\rangle = 1$ and no fringe visibility (complete decoherence) when $\langle P_0|P_1\rangle = 0$.

(a.) [1 Point] So far, we have considered the case where no infromation is acquired about the probe. Now we consider performing a measurement on the probe, but rather than considering a measurement of the basis $\{|+\rangle_P, |-\rangle_P\}$, as was done in the previous problem, we consider a measurement of the basis $|0\rangle_P, |1\rangle_P$. With what probability does each outcome occur in a measurement of the state $|\psi\rangle_{SP}$ as defined above?

(b.) [1 Point] For each outcome of this measurement, what state should be assigned to the photon path system after learning this outcome?

(c.) [2 Points] Determine the fringe visibility in each case.

(d.) [1 Point] At this point, you should see that for each outcome of the measurement on the probe, one recovers an interference pattern with maximum fringe visibility. Show that if one averages these interference patterns with their relative probabilities, one recovers the equations for $P(D0)$ and $P(D1)$ above, the interference pattern in the case where the probe is ignored.

You have now shown that an appropriate measurement on the probe allows one to recover the coherence in the system even in the limit where $\langle P_0|P_1\rangle = 0$.

The quantum eraser experiment shows that a system which has lost coherence relative to some basis may nonetheless be 'recohered' (and the which-way information effectively 'erased' from the environment) if one implements the appropriate measurement on the environment. Now, you will explore some constraints on an experimenters access to the environment under which such recoherence becomes impossible.
We consider the same experimental set-up as the quantum eraser experiment, but where there are now n distinct probes, each of which successively couples to the system by the unitary described above, leaving the composite of system and probes in the final state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_S |P_0\rangle_{P_1} ... |P_0\rangle_{P_n} + e^{i\phi} |1\rangle_S |P_1\rangle_{P_1} ... |P_1\rangle_{P_n}). \tag{14}$$

Now, imagine an observer that has access to only *one* of these n probes. This might occur, for instance, if the probes are photons that carry away information about the system to different regions of space and our observer is localized in only one of these many regions. It is generically the case for realistic environments, practical experiments typically have access to only a small part of the overall environment that is being interacted with. Suppose, for definiteness, that our experimenter has access to the nth probe. By symmetry, our conclusions will be independent of this choice.

(e.) [3 Points] Imagine that the experimenter measures the basis $\{|+\rangle, |-\rangle\}$ on the nth probe. The process of the nth probe becoming correlated with the photon (already part of an entangled state with the other n-1 probes), followed by measurement of the probe by the experimenter yields an effective measurement on the photon. What is the final state of the system after the measurement, for each of the two outcomes? What information does the experimenter have about the which-way observable of the photon, i.e. given the result of measurement of the nth probe, with what probability can the experimenter correctly predict the result of a measurement of this observable (if it were performed instead of the interference experiment)? Verify that this yields just as much which-way information as if there were just a single probe particle interacting with the system. We can therefore conclude that for the interaction considered here, which-way information about the photon is *redundantly encoded* in the environment.

(f.) [2 Points] Now imagine that the experimenter measures the basis $\{|0\rangle, |1\rangle\}$ on the nth probe. What is the final state of the system after the measurement, for each of the two outcomes? Show that in this case, the fringe visibility of the interference pattern can be obtained from the expression for the fringe visibility in the case where the probe is ignored by simply replacing $\langle P_0|P_1\rangle$ with $\langle P_0|P_1\rangle^{n-1}$.

(g.) [1 Point] Suppose $\langle P_0|P_1\rangle = 1 - \epsilon$ where ϵ is much smaller than 1. In this case, how large must n be before the fringe visibility becomes negligible? Suppose $\langle P_0|P_1\rangle = \epsilon$. How large must n be before the fringe visibility becomes negligible?

If one tries to define 'classical information' about a system as information about which there can be intersubjective agreement among many observers, then classical information is information that can be redundantly encoded in the different parts of the environment of the system. (This idea is called 'quantum Darwinism'.) The which-way information in our example was of this type. We have also seen that once the environment has achieved a redundant encoding of information about some basis of the system, then it is in practice infeasible for an experimenter who has access to only part of the environment to recover coherence for this basis.