Ph125c Spring 2017
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Homework - 4
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1 The Mach-Zehnder Interferometer [9 Points]

In the discussion of interference in textbooks, one typically considers the nature of the interference pattern when one does or does not learn about the path information. In this problem, we will consider a scenario wherein one can acquire partial path information. We will consider two cases: one wherein the experimenter acquires the path information, and the other wherein this information is encoded in the environment, but is not available to the experimenter.
A stripped-down version of the double-slit experiment is the Mach-Zehnder interferometer, shown in Figure 1.

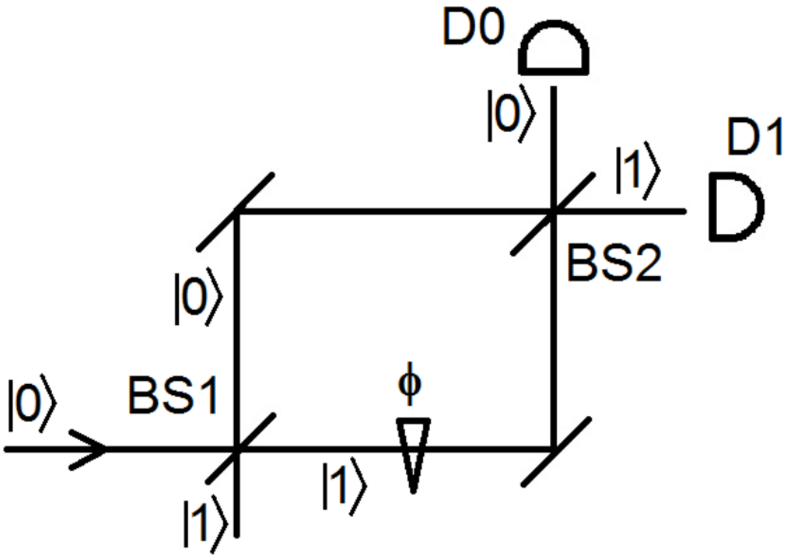

Figure 1: A Mach-Zehnder Interferometer

Photons are made to enter the interferometer one at a time, so there is at most one photon in the apparatus at any time; the path degree of freedom forms a two-level system labelled by $|0\rangle, |1\rangle$. We will call this the 'which-path' basis. $|0\rangle$ indicates that the photon is travelling in the upper arm of the interferometer, whereas $|1\rangle$ indicates that it is in the lower arm. The first beamsplitter, BS1, splits the input path $|0\rangle$ into a coherent superposition of two paths within the interferometer. A phase shifter placed in one of these paths allows the experimenter to change the relative phase $e^{i\phi}$ between $|0\rangle$ and $|1\rangle$ introduced between the arms of the interferometer, which are then recombined at a second beamsplitter BS2. Photons are detected by photodetectors D0, D1 in the output arms. If the input and output arms are labelled as in the figure, the unitary U_p describing each of the two (identical) beamsplitters and the unitary U_p describing the phase shifter are given by

$$U_b = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\langle 0| + \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\langle 1| \tag{1}$$

$$U_p = |0\rangle\langle 0| + e^{i\phi}|1\rangle\langle 1|. \tag{2}$$

The photodetectors make a measurement of how the photon exited the interferometer, and they measurement they perform are described by the PVM $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$. You should be able to convince yourself that for input in just one arm, corresponding to the input state $|0\rangle$, the probabilities of detection at D0, D1 respectively are given by

$$P(D0) = \frac{1}{2}(1 + \cos\phi) \tag{3}$$

$$P(D1) = \frac{1}{2}(1 - \cos\phi). \tag{4}$$

In an ideal interference experiment, the internal paths recombine coherently at BS2, and by choosing the phase appropriately, the experimenter can vary from total destructive to total constructive interference in a given output arm. This is what we see in the above equations. Note that the combination of the second beamsplitter plus detectors yields an effective measurement on the photon in the region between the beamsplitters (say, just before it reaches BS2) in a basis $|+\rangle = \{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\}$. That is, we can lump BS2 and the photodetectors into a single device, which measures the PVM $\{|+\rangle\langle +|, |-\rangle\langle -|\}$ inside the interferometer (after the photon has gone through the phase shifter, if it took the lower path). This basis is complementary to the basis containing path information. Decoherence in the 'which-path' ($|0\rangle$ and $|1\rangle$) basis has observable effects in the measurement statistics in this complementary basis. If the contributions from different internal paths become partially or fully incoherent, the visibility of the interference fringes (which is seen by doing repeated experiments and comparing the frequency of D0 clicks vs D1 clicks) is reduced, or the interference disappears altogether.

Now suppose we introduce a probe system that 'learns' partial path information in the following way. The probe interacts with the photon if it travels down path 1 (after the phase shifter). We treat the probe also as a quantum system, initially in a pure state $|P0\rangle$. As a result of the interaction between the photon and the probe, by the time the photon reaches BS2, the probe either remains in state $|P0\rangle$ (if the photon was going down path 0) or has evolved into state $|P1\rangle$ (if the photon was going down path 1). We can describe this coupling via an interaction unitary acting after BS1 and the phase shifter as follows:

$$U = (|0\rangle\langle 0|)_S \otimes I_P + (|1\rangle\langle 1|)_S \otimes (|P_1\rangle\langle P_0| + |P_1^\perp\rangle\langle P_0|^\perp)_P \tag{5}$$

where $|P_0^\perp\rangle$, $|P_1^\perp\rangle$ are states orthogonal to $|P0\rangle$, $|P1\rangle$ respectively, and we use the labels S, P to refer to the photon path system and probe, respectively, when it is not clear from the context to which system we are referring.

(a.) [2 Points] We first consider the case where the experimenter does not measure the environment. The interaction with the environment, even if the experimenter does not access it, causes a change in the measurement probabilities and the visibility of interference that one would see by changing ϕ. Assume input state $|0\rangle_S$ to the interferometer, as before. What is the state (i.e. the reduced density operator) describing the photon path degree of freedom before BS2? Write the state in the $|0\rangle_S$, $|1\rangle_S$ basis, as a function of the overlap $\langle P_0|P_1\rangle$ of the probe states; note the possibility for reduced coherence in this basis (the reduction of the off-diagonal elements of the density matrix) due – as we will argue below – to the partial path information now contained in the environment.

Solution: We start with the initial state $|0\rangle_S |P_0\rangle_P$ on the joint system of the photon (S) and the probe (P). After going through BS1, the state is

$$\begin{aligned}(U_b \otimes I)\, |0\rangle_S |P_0\rangle_P &= |+\rangle_S |P_0\rangle_P && (6)\\ &= \frac{1}{\sqrt{2}}(|0\rangle_S |P_0\rangle_P + |1\rangle_s |P_0\rangle_P) && (7)\end{aligned}$$

and after the phase shifter we have

$$(U_p \otimes I)\frac{1}{\sqrt{2}}(|0\rangle_S |P_0\rangle_P + |1\rangle_s |P_0\rangle_P) = \frac{1}{\sqrt{2}}(|0\rangle_S |P_0\rangle_P + e^{i\phi} |1\rangle_s |P_0\rangle_P) \qquad (8)$$

and finally after interacting with the probe via U, we have the state $|\psi\rangle$,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_S |P_0\rangle_P + e^{i\phi} |1\rangle_S |P_1\rangle_P). \qquad (9)$$

Note that whereas the state before U was factorable (the state on the photon system is just $\frac{1}{\sqrt{2}}(|0\rangle_S + e^{i\phi} |1\rangle_s)$), the state after is entangled unless $|P_1\rangle_P = |P_0\rangle_P$ up to some phase. Thus, in general, the state on S will be mixed and need to be calculated as a reduced density matrix. The density matrix of the photon and probe together is

$$\rho_{SP} = |\psi\rangle \langle\psi| \qquad (10)$$

and we find the state on S by taking the partial trace over the probe system:

$$\begin{aligned}\rho_S &= \mathrm{Tr}_P\,(|\psi\rangle \langle\psi|) && (11)\\ &= \mathrm{Tr}_P \left(\frac{1}{2}(|0\rangle_S |P_0\rangle_P + e^{i\phi} |1\rangle_S |P_1\rangle_P)(\langle 0|_S \langle P_0|_P + e^{-i\phi} \langle 1|_S \langle P_1|_P)\right) && (12)\\ &= \frac{1}{2}\mathrm{Tr}_P(|0\rangle_S |P_0\rangle_P \langle 0|_S \langle P_0|_P + e^{i\phi} |1\rangle_S |P_1\rangle_P \langle 0|_S \langle P_0|_P && (13)\\ &+ e^{-i\phi} |0\rangle_S |P_0\rangle_P \langle 1|_S \langle P_1|_P + |1\rangle_S |P_1\rangle_P \langle 1|_S \langle P_1|_P) && (14)\\ &= \frac{1}{2}(\mathrm{Tr}_P\,(|0\rangle_S |P_0\rangle_P \langle 0|_S \langle P_0|_P) + e^{i\phi}\mathrm{Tr}_P\,(|1\rangle_S |P_1\rangle_P \langle 0|_S \langle P_0|_P) && (15)\\ &+ e^{-i\phi}\mathrm{Tr}_P\,(|0\rangle_S |P_0\rangle_P \langle 1|_S \langle P_1|_P) + \mathrm{Tr}_P\,(|1\rangle_S |P_1\rangle_P \langle 1|_S \langle P_1|_P)). && (16)\end{aligned}$$

Now, note that performing a partial trace over P requires choosing an orthonormal basis for P, which $\{|P_0\rangle, |P_1\rangle\}$ is not. However, $\{|P_0\rangle, |P_0^\perp\rangle\}$ and $\{|P_1\rangle, |P_1^\perp\rangle\}$ are. Using the first basis to perform the first partial trace, and the second basis to perform the last partial trace yields

$$\begin{aligned}\mathrm{Tr}_P\,(|0\rangle_S |P_0\rangle_P \langle 0|_S \langle P_0|_P) &= |0\rangle \langle 0|_S && (17)\\ \mathrm{Tr}_P\,(|1\rangle_S |P_1\rangle_P \langle 1|_S \langle P_1|_P) &= |1\rangle \langle 1|_S && (18)\\ & && (19)\end{aligned}$$

and for the middle two partial traces, using either basis will yield the following results

$$\begin{aligned}\mathrm{Tr}_P\,(|1\rangle_S |P_1\rangle_P \langle 0|_S \langle P_0|_P) &= \langle P_0|P_1\rangle\, |1\rangle \langle 0|_S && (20)\\ \mathrm{Tr}_P\,(|0\rangle_S |P_0\rangle_P \langle 1|_S \langle P_1|_P) &= \langle P_0|P_1\rangle^* |0\rangle \langle 1|_S\,. && (21)\end{aligned}$$

Thus our reduced density matrix on the photon system S is

$$\rho_S = \frac{1}{2}\Big(|0\rangle\langle 0|_S + |1\rangle\langle 1|_S + e^{i\phi}\langle P_0|P_1\rangle\,|1\rangle\langle 0|_S + e^{-i\phi}\langle P_0|P_1\rangle^*\,|0\rangle\langle 1|_S\Big) \tag{22}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & e^{-i\phi}\langle P_0|P_1\rangle^* \\ e^{i\phi}\langle P_0|P_1\rangle & 1 \end{pmatrix}. \tag{23}$$

Now, let's make a few conceptual points about the result: suppose that we had $|P_0\rangle = c\,|P_1\rangle$ where c is some magnitude-one complex number (that is, the probe states are co-linear). Then the the reduced state is $\rho_S = \frac{1}{2}\left(|0\rangle\langle 0|_S + |1\rangle\langle 1|_S + ce^{i\phi}\langle P_0|P_1\rangle\,|1\rangle\langle 0|_S + c^*e^{-i\phi}\langle P_0|P_1\rangle^*\,|0\rangle\langle 1|_S\right)$, which actually a pure state: $\rho_S = \frac{1}{\sqrt{2}}\left(|0\rangle + ce^{i\phi}\,|1\rangle\right)\frac{1}{\sqrt{2}}\left(\langle 0| + c^*e^{-i\phi}\,\langle 1|\right)$. So, in this case, the state of the photon is a pure quantum state which is a coherent quantum superposition between having taken the upper path of the interferometer $|0\rangle$ and having taken the lower path of the interferometer $|1\rangle$ and picking up the relative phase $ce^{i\phi}$. But, if we now start varying $|P_0\rangle$ so that it's no longer co-linear to $|P_1\rangle$, as they become closer and closer to orthogonal, the magnitue of the offdiagonal entries, $|\langle P_0|P_1\rangle|$, gets closer and closer to zero (and is zero in the orthogonal limit). If $|\langle P_0|P_1\rangle|$ is less than 1, the state on the photon system S is no longer pure, which means that there's some basis in which the reduced density matrix will be diagonal, and there will be more than one non-zero diagonal element, indicating that the state is now a statistical mixture of different orthogonal pure states with different probabilities. As $|\langle P_0|P_1\rangle|$ shrinks, this probability distribution will become noisier and noisier, until we reach the limit of $|\langle P_0|P_1\rangle| = 0$, in which case the density matrix is

$$\rho_S = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \frac{1}{2}\,|0\rangle\langle 0| + \frac{1}{2}\,|1\rangle\langle 1|\,. \tag{24}$$

which is the maximally mixed state, a 50/50 statistical mixture, which tells you that the system is *either* the pure state $|0\rangle\langle 0|$ *or* the distinct orthogonal pure state $|1\rangle\langle 1|$, with equal probabilities. This is conceptually completely different than a quantum superposition of the two pure states of the form $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, which is its own distinct pure quantum state. Notice that as $|\langle P_0|P_1\rangle|$ shrinks, there is less and less dependence of the reduced density matrix on the relative phase $e^{i\phi}$. When the reduced density matrix is pure, we can observe this relative phase by doing an interference experiment, which is what we explore in the next problem. We will see that our ability to observe interference between the $|0\rangle$ and $|1\rangle$ parts of the quantum state in coherent superposition reduces as the photon system becomes more entangled with the probe system. This is because of the phenomenon of *decoherence* as discussed in lecture.

(b.) [1 Point] Find the probability of getting a click in detector D0, as a function of ϕ and the overlap $\langle P_0|P_1\rangle$. This causes visibility of interference to be reduced (as seen in the next problem).

Solution: We know that the state of the photon interacting with BS2 (unitary U_b) and then the detectors (which measure PVM $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$) is equivalent to simply performing the PVM $\{|+\rangle\langle +|, |-\rangle\langle -|\}$ on the state of the photon system just before BS2 (indeed, observe that $U_b\,|0\rangle\langle 0|\,U_b^\dagger = |+\rangle\langle +|$ and $U_b\,|1\rangle\langle 1|\,U_b^\dagger = |-\rangle\langle -|$). So, performing this measurement, we can calculate the measurement probabilities via the Born rule:

$$\Pr(D0) = \Pr(+) = \mathrm{Tr}\left(|+\rangle\langle +|\,\rho_S\right) \tag{25}$$

$$= \mathrm{Tr}\left(|+\rangle\langle +|\,\frac{1}{2}\Big(|0\rangle\langle 0|_S + |1\rangle\langle 1|_S + e^{i\phi}\langle P_0|P_1\rangle\,|1\rangle\langle 0|_S + e^{-i\phi}\langle P_0|P_1\rangle^*\,|0\rangle\langle 1|_S\Big)\right)$$

$$
\begin{aligned}
&= \frac{1}{2}\Big(\langle 0|+\rangle\langle +|0\rangle + e^{i\phi}\langle P_0|P_1\rangle\langle 0|+\rangle\langle +|1\rangle + e^{-i\phi}\langle P_0|P_1\rangle^*\langle 1|+\rangle\langle +|0\rangle + \langle 1|+\rangle\langle +|1\rangle\Big) \\
&= \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}(e^{i\phi}\langle P_0|P_1\rangle + e^{-i\phi}\langle P_0|P_1\rangle^*) + \frac{1}{2}\right) \qquad (26)\\
&= \frac{1}{2}\Big(1 + \mathrm{Re}(e^{i\phi}\langle P_0|P_1\rangle)\Big). \qquad (27)
\end{aligned}
$$

A similar calculation gives

$$
\Pr(D1) = \Pr(-) = \frac{1}{2}\Big(1 - \mathrm{Re}(e^{i\phi}\langle P_0|P_1\rangle)\Big). \qquad (28)
$$

The probabilities of either detector clicking depend on ϕ, the relative phase that the photon picks up if it travels down the lower arm of the interferometer. This combination of BS2 and the detectors (which abstractly is equivalent to the PVM $\{|+\rangle\langle +|, |-\rangle\langle -|\}$ being measured on the photon state inside the interferometer) allows to observe *quantum interference* between the two possible paths the photon takes in quantum superposition. Let's see exactly why, by considering the case where there's no probe, or $\langle P_0|P_1\rangle = 1$. Then the probabilities are $\Pr(D0) = \frac{1}{2}(1 + \cos\phi)$ and $\Pr(D1) = \frac{1}{2}(1 - \cos\phi)$, which can be tuned to be *any* two complementary probabilities we want by changing ϕ appropriately. What's the quantum mechanical mechanism for this? Before BS2, the photon state is in quantum superposition of the two paths $\frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi}|1\rangle)$ where if we measured which path it took without having BS2 in the system, we would just get that with 50 percent probability it took the upper path and 50 percent probability the lower path. There's no observable effect of the relative phase ϕ between the two parts of the superposition. However, the role of BS2 is to actually show that there is a measurable physical effect to the relative phase in the quantum superposition: it turns $\frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi}|1\rangle)$ into

$$
\frac{1}{\sqrt{2}}(|+\rangle + e^{i\phi}|-\rangle) = \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) + e^{i\phi}\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)) = \frac{1}{\sqrt{2}}((1 + e^{i\phi})|0\rangle + (1 - e^{i\phi})|1\rangle) \qquad (29)
$$

so, now adjusting ϕ adjusts the relative probability amplitudes for the $|1\rangle$ path and the $|0\rangle$ path. The second beam splitter BS2 is causing the two parts of the quantum state which represent the two paths in the interferometer to *interfere constructively or destructively*, depending on ϕ. This is a hallmark *quantum* phenomenon that is not seen in classical systems!

(c.) [3 Points] Above, we were considering the case where the experimenter directly measures the internal system of the interferometer (where the photon is in some quantum superposition of travelling down either path), with the PVM $\{|+\rangle\langle +|, |-\rangle\langle -|\}$ in order to observe interference between the two paths which can be modulated by changing ϕ. Now consider the case in which the experimenter measures the probe system while the photon is in the interferometer (after interacting with the probe, but before reaching BS2) in order to try and infer what path of the interferometer the photon is in (the goal of this measurement is NOT to observe interference, as in the previous part). Since we are not concerned with interference in this part, and all measurements are occuring before the photon hits BS2, you can pretend that BS2 and the photodetectors do not exist in this part of the problem. It is useful to introduce an orthonormal basis $|0\rangle_P$, $|1\rangle_P$ for the probe system, such that the probe states may be written

$$
|P_0\rangle = \cos\theta\,|0\rangle_P + \sin\theta\,|1\rangle_P \qquad (30)
$$

$$
|P_1\rangle = \cos\theta\,|0\rangle_P - \sin\theta\,|1\rangle_P \qquad (31)
$$

where 2θ is the real angle parameterizing the overlap between the probe states: $\langle P_0|P_1\rangle = \cos(2\theta)$. It may be shown (although you are not being asked to) that the measurement which optimally discriminates between the non-orthogonal probe states, in the sense of minimizing the error in guessing the state, is then a measurement in the $|\pm\rangle_P = \frac{1}{\sqrt{2}}(|0\rangle_P \pm |1\rangle_P)$ basis. If the experimenter does measure the probe with this optimal measurement, he acquires information about the which-way observable (which branch of the interferometer the photon is in). How much information is gained may be quantified in terms of his probability of successfully predicting which path the photon is actually in, based on his probe measurement results. The experimenter's measurement on the probe system is described by the PVM $\{Q_0 = |+\rangle\langle +|_P, Q_1 = |-\rangle\langle -|_P\}$, where outcome i is taken to imply the corresponding outcome i in a which-way measurement. Show that the experimentalist's total probability of success in inferring the path the photon took based on the probe measurement readout is $\frac{1}{2}(1+\sin(2\theta))$. This shows that as the probe states become closer to orthogonal, the experimentalist can become near certain of which path the photon takes in the interferometer (and thus acquiring more information about this observable via the probe).

Solution: Now, instead of trying to observe interference between the paths of the interferometer as in the above, we are going to do a measurement of the probe while the photon is inside the interferometer (before hitting BS2) and use its result to infer what branch of the interferometer we would find the photon in if we measured it directly (that measurement would just be the PVM $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ on the internal photon state, no BS2). The joint state of the photon and probe system $S\otimes P$ before BS2 is $\frac{1}{\sqrt{2}}\left(|0\rangle|P_0\rangle + e^{i\phi}|1\rangle|P_1\rangle\right)$. This is a pure state, and we're going to measure a PVM $\{I_S\otimes|+\rangle\langle +|_P, I_S\otimes|-\rangle\langle -|_P\}$. If we get measurement outcome $+$ on the probe, we will assume that if we measured which branch of the interferometer the photon is in, we would definitely get $|0\rangle$, and for outcome $-$ then branch $|1\rangle$. But we won't always be right. The actual post-measurement state will end up being a pure product state which will generally be some superposition of $|0\rangle$ and $|1\rangle$ for the photon system, and just the state corresponding to the measurement outcome $|\pm\rangle$ on the probe system. If we wanted to know the probabilities with which our inference of the photon path is correct, we could measure the probabilities of measuring 0 or 1 on the photon system after getting $+$ or $-$ on the probe system, respectively. Since the PVM $\{I_S\otimes|+\rangle\langle +|_P, I_S\otimes|-\rangle\langle -|_P\}$ on the probe and PVM $\{|0\rangle\langle 0|_S\otimes I_P, |1\rangle\langle 1|_S\otimes I_P\}$ on the photon path commute, it doesn't matter which order they're measure in, and can thus be combined into a single joint PVM on the joint system for the purpose of calculating measurement statistics. So, we'll calculate the probabilities of getting the outcomes $(0,+)$ and $(1,-)$ for the PVM $\{|0\rangle\langle 0|_S\otimes|+\rangle\langle +|_P, |0\rangle\langle 0|_S\otimes|-\rangle\langle -|_P, |1\rangle\langle 1|_S\otimes|+\rangle\langle +|_P, |1\rangle\langle 1|_S\otimes|-\rangle\langle -|_P\}$. Since we're now measuring a PVM on a pure state, we can just compute the overlaps to get the probabilities. To do the calculation, it will be useful to observe that $\langle P_0|\pm\rangle = \frac{1}{\sqrt{2}}(\cos\theta \pm \sin\theta)$, and $\langle P_1|\pm\rangle = \frac{1}{\sqrt{2}}(\cos\theta \mp \sin\theta)$. The probability that we measure $+$ on the probe and are able to correctly infer that the photon is in the 0 branch is

$$\begin{aligned}
\Pr(0,+) &= |\langle 0|\langle +|\frac{1}{\sqrt{2}}\left(|0\rangle|P_0\rangle + e^{i\phi}|1\rangle|P_1\rangle\right)|^2 && (32)\\
&= \frac{1}{2}|\langle +|P_0\rangle|^2 && (33)\\
&= \frac{1}{4}(\cos\theta + \sin\theta)^2 && (34)\\
&= \frac{1}{4}(1+\sin(2\theta)). && (35)
\end{aligned}$$

By a nearly identical calculation, the probability of getting measurement outcome $-$ on the probe and correctly inferring that the photon is in the 1 branch is

$$\Pr(1,-) \quad = \quad \frac{1}{4}(1+\sin(2\theta)). \tag{36}$$

Thus the total probability of success in the task of using a measurement on the probe to guess which branch of the interferometer the photon took is $\Pr(0,+) + \Pr(1,-) = \frac{1}{2}(1+\sin(2\theta))$. What is the significance of this result? Well, looking back at our results from problem b and using the θ parameterization of the probe states, we see that the probabilities of each detector firing in an interference experiment (where there is a probe present, but it isn't measured by the experimentalist) are

$$\Pr(D0) \quad = \quad \frac{1}{2}\left(1+\cos(2\theta)\cos\phi\right) \tag{37}$$

$$\Pr(D1) \quad = \quad \frac{1}{2}\left(1-\cos(2\theta)\cos\phi\right). \tag{38}$$

If $\cos(2\theta)$ is not equal to 1 or -1 (i.e. if the probe states are not co-linear), then the effect that changing ϕ has on the outcome probabilities in the interference experiment is diminished, and as the probe states become more and more orthogonal, the two probabilities become closer and closer to 1/2 and changes to them by changing ϕ become harder and harder to measure. So, as the probe states are becoming more orthogonal, the observability of quantum interference in the interference experiment diminishes until changing ϕ has absolutely no effect at all (i.e. there is no longer any interference between the branches of the interferometer)! Why is this?? Well, the suggestion that we're making here is this. You calculated that *if* you measured the probe, you would be able to correctly guess where the photon is in the interferometer with probability $\frac{1}{2}(1+\sin(2\theta))$. As the probe states become more and more orthogonal, this probability tends to 1. That means as the probe states become more and more orthogonal, more and more exact information about where the photon is in the interferometer exists in the probe system (and could in principle be extracted if the experimentalist measured the probe properly). So what we're seeing is that as more and more information about where the photon is exists in the probe, the less we are able to demonstrate the quantum phenomenon of interference between to the two branches of the interferometer! As we saw in part a, the interaction of the photon with the probe *decoheres* the state of the photon, and in the orthogonal probe state limit, the state of the photon in the interferometer is no longer in a coherent quantum superposition, but a classical probabilistic mixed state. So, we're seeing that decoherence corresponds directly to a loss of *quantum* phenomena, hence our system behaves less quantumly and more classically as it decoheres!!

(d.) [3 Points] The combination of interaction with the probe, followed by measurement of the probe, as we have seen previously, can be described as a measurement on the photon path system alone (much in the same way as was done in problem 1a of homework 3). Find the 1-parameter family of POVMs (parameterized by θ) describing the effective measurement on the system. Comment on the limits where $|P_0\rangle$ and $|P_1\rangle$ are (i.) co-linear, (ii.) orthogonal.

Solution: We can think of this setup as being a von Neumann measurement setup, where what we would like to do is measure which path the photon is in (that is, just before BS2), and instead of doing this measurement directly on the photon system, we will instead introduce a measurement apparatus (the probe in initial state $|P_0\rangle$), entangle it with the system of interest (with interaction unitary U as given in the problem), and then perform a PVM on the measurement apparatus (probe) in order to get information about what is going on in the photon system. As we know, and saw on the last homework, this indirect von Neumann measurement (which we explored in the previous part of the problem) can also be modelled as a direct POVM measurement on the photon system. The way we do that is by realizing (as in the last homework) that a POVM on system S with POVM elements

$$E_+ = \mathrm{Tr}_P \left(I_S \otimes |P_0\rangle \langle P_0| U^\dagger I_S \otimes |+\rangle \langle +| U \right) \tag{39}$$

$$E_- = \mathrm{Tr}_P \left(I_S \otimes |P_0\rangle \langle P_0| U^\dagger I_S \otimes |-\rangle \langle -| U \right) \tag{40}$$

obtains the same measurement statistics as the indirect von Neumann measurement on the probe. Plugging in all of the expressions and taking the traces, making use of the inner products $\langle P_0|\pm\rangle = \frac{1}{\sqrt{2}}(\cos\theta \pm \sin\theta)$, and $\langle P_1|\pm\rangle = \frac{1}{\sqrt{2}}(\cos\theta \mp \sin\theta)$, one finds that

$$E_+ = \frac{1+\sin(2\theta)}{2} |0\rangle \langle 0| + \frac{1-\sin(2\theta)}{2} |1\rangle \langle 1| \tag{41}$$

$$E_- = \frac{1-\sin(2\theta)}{2} |0\rangle \langle 0| + \frac{1+\sin(2\theta)}{2} |1\rangle \langle 1| . \tag{42}$$

This confirms what we already calculated in the previous part of the problem: if we got outcome $+$ for this POVM (which corresponds to having gotten outcome $+$ for the probe measurement), the photon path state would collapse to $|0\rangle$ with probability $\frac{1+\sin(2\theta)}{2}$ and if we got outcome $-$ for this POVM the photon path state would collapse to $|1\rangle$ with probability $\frac{1+\sin(2\theta)}{2}$. The $\frac{1-\sin(2\theta)}{2}$ coefficients are the complementary probabilities for the failure cases of the inferrence process discussed in part c. In the limit that the probe states are co-linear ($\theta = 0$), the POVM is actually the maximally mixed POVM

$$E_+ = \frac{1}{2} |0\rangle \langle 0| + \frac{1}{2} |1\rangle \langle 1| = \frac{I}{2} \tag{43}$$

$$E_- = \frac{1}{2} |0\rangle \langle 0| + \frac{1}{2} |1\rangle \langle 1| = \frac{I}{2} \tag{44}$$

which agrees with our understanding that if the probe states are colinear, then the probe doesn't get entangled with the photon system at all, and measuring the probe tells us absolutely nothing about the photon system. Both outcomes are equally likely, and the photon system doesn't collapse or change after the measurement. On the other hand in the limit that the probe states are orthogonal ($\theta = \pi/4$), we actually obtain a PVM

$$E_+ = |0\rangle \langle 0| \tag{45}$$

$$E_- = |1\rangle \langle 1| \tag{46}$$

which agrees with our understanding that if the probe states are orthogonal, the probe becomes maximally entangled with the photon system, and so measuring the probe system gives us perfect information about the photon path, as would a perfect PVM measurement $\{|0\rangle \langle 0|, |1\rangle \langle 1|\}$ directly on the photon system itself.

Here's the punchline: in part (b.), you see that the mere presence of the probe, which isn't even being measured but the photon interacts with, changes the experimenter's ability to observe interference between the paths of the interferometer (which he would do by measuring $\{|+\rangle \langle +|, |-\rangle \langle -|\}$ on the photon). As the probe states become closer to orthogonal, the ability of changing ϕ to change the relative measurement probabilities decreases, hence the experimenter's ability to cause interference between the two different path possibilities in the interferometer is being reduced. In part (c.), we have seen that as the probe states $|P_0\rangle$, $|P_1\rangle$ become more distinguishable (closer to orthogonal), the environment contains more information about the which-way observable (in the sense that if the experimenter chooses to access to it by measurement, they can predict outcomes of measurement of this observable correctly with higher probability). So, we conclude that as the environment learns more information about some basis of the system (even if the experimenter does not access it), the evolution of the system changes from unitary, coherent evolution on the one hand, to decoherent evolution on the other.

2 The Quantum Eraser [11 Points]

The decoherence that a system experiences by virtue of its interaction with the environment is often considered to be the explanation of the emergence of classicality for the system. For instance, it is decoherence that ensures that orthogonal states of a pointer on an apparatus do not interfere. Such decoherence is important for understanding how the collapse postulate for measurements can be recovered when the apparatus is treated quantum mechanically, as is done in the de Broglie-Bohm interpretation, dynamical collapse models and the Everett interpretation (some of which you may hear about in the next few weeks).

In the previous problem, we saw that the amount of coherence a system exhibits relative to some basis decreases as one increases the amount of information the environment has about that basis. In this question, you will show that by measuring different observables on the environment, one can obtain information about different observables on the system. In particular, we will reconsider the Mach-Zehnder interferometer experiment discussed in the previous problem. We will see that when the photon and the probe become entangled, the probe can simultaneously carry information about the which-way observable and the complementary observable (relative phase) of the photon, and that interference is recovered if one chooses to measure the latter. This is known as the 'quantum eraser' experiment.

We will also consider the case wherein experimenters only have access to a small part of a large environment and we will see that in this case, the loss of coherence for the system can become effectively irreversible. In other words, we will express the condition for irreversibility in terms of how information about the system is encoded in the environment.

Recall The Mach-Zehnder interferometer that you became acquainted with in problem 1. After BS1 and the phase-shifter, the photon is in state $\frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi}|1\rangle)$. A probe that starts out in the state $|P_0\rangle$ is, again, coupled to the photon via the unitary

$$U = (|0\rangle\langle 0|)_S \otimes I_P + (|1\rangle\langle 1|)_S \otimes (|P_1\rangle\langle P_0| + |P_1^{\perp}\rangle\langle P_0|^{\perp})_P \tag{47}$$

such that the state of the photon and probe after the interaction (but prior to BS2) is

$$|\Psi\rangle_{SP} = \frac{1}{\sqrt{2}}(|0\rangle_S|P_0\rangle_P + e^{i\phi}|1\rangle_S|P_1\rangle_P). \tag{48}$$

We will again assume that the probe states have the same form as in problem 1 (c.) in terms of the real parameter θ. We presume that after this interaction, the photon passes through BS2 and we register which detector fires. The experiment is repeated a large number of times, such that we obtain the relative frequency of the two detectors firing. As saw in the previous problem, the probability for each outcome, as a function of the relative phase ϕ between the arms of the intereferometer is

$$P(D0) = \frac{1}{2}(1 + \mathrm{Re}(e^{i\phi}\langle P_0|P_1\rangle)) = \frac{1}{2}(1 + \langle P_0|P_1\rangle\cos\phi) \tag{49}$$

$$P(D1) = \frac{1}{2}(1 - \mathrm{Re}(e^{i\phi}\langle P_0|P_1\rangle)) = \frac{1}{2}(1 - \langle P_0|P_1\rangle \cos\phi). \tag{50}$$

We define the fringe visibility V as

$$V = \frac{P_{max} - P_{min}}{P_{max} + P_{min}}, \tag{51}$$

where $P_{max} = \max_\phi P(D0)$, and similarly for P_{min}. We see that the fringe visibility in the presence of the probe system is given by

$$V = \langle P_0|P_1\rangle. \tag{52}$$

We have the maximum value of fringe visibility (perfect interference) when $\langle P_0|P_1\rangle = 1$ and no fringe visibility (complete decoherence) when $\langle P_0|P_1\rangle = 0$.

(a.) [1 Point] So far, we have considered the case where no infromation is acquired about the probe. Now we consider performing a measurement on the probe, but rather than considering a measurement of the basis $\{|+\rangle_P, |-\rangle_P\}$, as was done in the previous problem, we consider a measurement of the basis $|0\rangle_P, |1\rangle_P$. With what probability does each outcome occur in a measurement of the state $|\psi\rangle_{SP}$ as defined above?

Solution: The probability of measuring 0 is

$$\begin{aligned}
\Pr(0) &= \mathrm{Tr}\left(|\Psi\rangle\langle\Psi|_{SP}\, I_S \otimes |0\rangle\langle 0|_P\right) && (53)\\
&= \frac{1}{2}(|\langle P_0|0\rangle|^2 + |\langle P_1|0\rangle|^2) && (54)\\
&= \cos^2\theta && (55)\\
&= \frac{1}{2}(1 + \cos(2\theta)) && (56)\\
&= \frac{1}{2}(1 + \langle P_0|P_1\rangle) && (57)
\end{aligned}$$

and the probability of measuring 1 is

$$\begin{aligned}
\Pr(1) &= \mathrm{Tr}\left(|\Psi\rangle\langle\Psi|_{SP}\, I_S \otimes |1\rangle\langle 1|_P\right) && (58)\\
&= \frac{1}{2}(|\langle P_0|1\rangle|^2 + |\langle P_1|1\rangle|^2) && (59)\\
&= \sin^2\theta && (60)\\
&= \frac{1}{2}(1 - \cos(2\theta)) && (61)\\
&= \frac{1}{2}(1 - \langle P_0|P_1\rangle). && (62)
\end{aligned}$$

(b.) [1 Point] For each outcome of this measurement, what state should be assigned to the photon path system after learning this outcome?

Solution: Here since we're doing a projective measurement $\{I_S \otimes |0\rangle\langle 0|_P, I_S \otimes |1\rangle\langle 1|_P\}$ on a pure state $|\Psi\rangle_S P$ and we know the outcome probabilities, we can use the standard state update rule. For outcome 0 we have

$$|\Psi\rangle \rightarrow \qquad \frac{1}{\sqrt{\Pr(0)}} I_S \otimes |0\rangle\langle 0|_P |\Psi\rangle \tag{63}$$

$$
\begin{aligned}
&= \frac{1}{\cos\theta}\frac{1}{\sqrt{2}}\left(\langle 0|P_0\rangle\,|0\rangle_S + e^{i\phi}\langle 0|P_1\rangle\,|1\rangle_S\right)|0\rangle_P && (64)\\
&= \frac{1}{\cos\theta}\frac{1}{\sqrt{2}}\left(\cos\theta\,|0\rangle_S + e^{i\phi}\cos\theta\,|1\rangle_S\right)|0\rangle_P && (65)\\
&= \frac{1}{\sqrt{2}}(|0\rangle_S + e^{i\phi}|1\rangle_S)\,|0\rangle_P && (66)\\
&= |\Phi_+\rangle && (67)
\end{aligned}
$$

and a very similar calculation for outcome 1 yields

$$
|\Psi\rangle \to \frac{1}{\sqrt{2}}(|0\rangle_S - e^{i\phi}|1\rangle_S)\,|1\rangle_P = |\Phi_-\rangle\,. \tag{68}
$$

(c.) [2 Points] Determine the fringe visibility in each case.

Solution: Okay, so, just to recap what we're doing: the beam has gone into the interferometer, and interacted with the probe. In the previous problem, we saw that in general if we ignore the probe, decoherence caused by the probe will cause us to lose the ability to see quantum interference (aability measured by the quantity $V = \langle P_0|P_1\rangle$). What is also true (but I didn't ask you to show), is that if you do a probe measurement in the $+/-$ basis as in problem 1, and then try to do an interference experiment using BS2 and the detectors afterwards, you won't see any interference: the $+/-$ measurement collapses the state of the photon in the interferometer to collapse to just $|0\rangle$ or just $|1\rangle$, with no sign of the phase ϕ, so there will be no interference. Now we ask, what if we measure the probe in the basis $|0\rangle, |1\rangle$ with the PVM $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ and then try and do the interference experiment with BS2 and the detectors? Will the situation be any different than if we measured the probe in $+/-$ and tried it (which would result in no interference)? Let's find out!

If we got measurement outcome 0 on the probe, then the state of the system and probe is $|\Phi_+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_S + e^{i\phi}|1\rangle_S)\,|0\rangle_P$, so the state on just the photon system is actually a pure state $\frac{1}{\sqrt{2}}(|0\rangle_S + e^{i\phi}|1\rangle_S)$! Letting it hit BS2, we now have

$$
\begin{aligned}
U_b\frac{1}{\sqrt{2}}(|0\rangle_S + e^{i\phi}|1\rangle_S) &= \frac{1}{\sqrt{2}}(|+\rangle_S + e^{i\phi}|-\rangle_S) && (69)\\
&= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0\rangle_S + |1\rangle_S) + e^{i\phi}\frac{1}{\sqrt{2}}(|0\rangle_S - |1\rangle_S)\right) && (70)\\
&= \frac{1+e^{i\phi}}{2}|0\rangle_S + \frac{1-e^{i\phi}}{2}|1\rangle_S && (71)
\end{aligned}
$$

and we calculate $\Pr(D0) = \frac{1+\cos\phi}{2}$. Thus $V = 1$, which is telling us that by adjusting ϕ we have complete control over the measurement statistics of either detector, meaning we have regained the ability to have perfect quantum interference!

Similarly, in the case that we got outcome 1 on the probe, then the state of the system and probe is $|\Phi_-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_S - e^{i\phi}|1\rangle_S)\,|1\rangle_P$, so the state on just the photon system is actually a pure state $\frac{1}{\sqrt{2}}(|0\rangle_S - e^{i\phi}|1\rangle_S)$! Letting it hit BS2, we now have

$$
U_b\frac{1}{\sqrt{2}}(|0\rangle_S - e^{i\phi}|1\rangle_S) = \frac{1}{\sqrt{2}}(|+\rangle_S - e^{i\phi}|-\rangle_S) \tag{72}
$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0\rangle_S + |1\rangle_S) - e^{i\phi}\frac{1}{\sqrt{2}}(|0\rangle_S - |1\rangle_S)\right) \quad (73)$$

$$= \frac{1-e^{i\phi}}{2}|0\rangle_S + \frac{1+e^{i\phi}}{2}|1\rangle_S \quad (74)$$

and we calculate $\Pr(D0) = \frac{1-\cos\phi}{2}$. Thus $V = 1$, which, again, is telling us that by adjusting ϕ we have complete control over the measurement statistics of either detector, meaning we have regained the ability to have perfect quantum interference!

The significance of the result is as follows: we saw in the previous problem that the fact of the probe existing causes decoherence in our interferometer system. The mechanism is that as the probe learns more and more information about the path the photon took, the quantum state of the photon becomes less and less quantum. So we can imagine that information about the photon is leaking from the photon system into the probe system, and the more that this happens, the more the photon system decoheres. What we've shown here is that if we have access to the probe, where the photon's path information has gone, we can reverse this leakage of information by making the appropriate measurement, which reverses the decoherence and allows us to regain the quantum properties of the photon system (in this case quantum interference). Or another way to say it, is that making the $|0\rangle, |1\rangle$ measurement on the probe *erases* the information about the photon system from the probe, which is what allows us to regain coherence in the photon system.

(d.) [1 Point] At this point, you should see that for each outcome of the measurement on the probe, one recovers an interference pattern with maximum fringe visibility. Show that if one averages these interference patterns with their relative probabilities, one recovers the equations for $P(D0)$ and $P(D1)$ above, the interference pattern in the case where the probe is ignored.

Solution: Now, what we're doing can be conceptually interpreted as having a measurement in the 0/1 basis being done on the probe, but not looking at the outcome. We now want to see what $\Pr(D0)$ would be in an interference experiment where this happens. Using the laws of conditional probability, we see that

$$\Pr(D0) = \Pr(0)\Pr(D0|0) + \Pr(1)\Pr(D0|1) \quad (75)$$

$$= \frac{1+\cos(2\theta)}{2}\frac{1+\cos\phi}{2} + \frac{1-\cos(2\theta)}{2}\frac{1-\cos\phi}{2} \quad (76)$$

$$= \frac{1+\cos(2\theta)\cos\phi}{2} \quad (77)$$

so this has the effect of basically ignoring the probe altogether, which is what we did in problem 1b, which is why the probabilities match.

You have now shown that an appropriate measurement on the probe allows one to recover the coherence in the system even in the limit where $\langle P_0|P_1\rangle = 0$.

The quantum eraser experiment shows that a system which has lost coherence relative to some basis may nonetheless be 'recohered' (and the which-way information effectively 'erased' from the environment) if one implements the appropriate measurement on the environment. Now, you will explore some constraints on an experimenters access to the environment under which such recoherence becomes impossible.

We consider the same experimental set-up as the quantum eraser experiment, but where there are now n distinct probes, each of which successively couples to the system by the unitary described above, leaving the composite of system and probes in the final state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_S |P_0\rangle_{P_1} ... |P_0\rangle_{P_n} + e^{i\phi} |1\rangle_S |P_1\rangle_{P_1} ... |P_1\rangle_{P_n}). \tag{78}$$

Now, imagine an observer that has access to only *one* of these n probes. This might occur, for instance, if the probes are photons that carry away information about the system to different regions of space and our observer is localized in only one of these many regions. It is generically the case for realistic environments, practical experiments typically have access to only a small part of the overall environment that is being interacted with. Suppose, for definiteness, that our experimenter has access to the nth probe. By symmetry, our conclusions will be independent of this choice.

(e.) [3 Points] Imagine that the experimenter measures the basis $\{|+\rangle, |-\rangle\}$ on the nth probe. The process of the nth probe becoming correlated with the photon (already part of an entangled state with the other n-1 probes), followed by measurement of the probe by the experimenter yields an effective measurement on the photon. What is the final state of the system after the measurement, for each of the two outcomes? What information does the experimenter have about the which-way observable of the photon, i.e. given the result of measurement of the nth probe, with what probability can the experimenter correctly predict the result of a measurement of this observable (if it were performed instead of the interference experiment)? Verify that this yields just as much which-way information as if there were just a single probe particle interacting with the system. We can therefore conclude that for the interaction considered here, which-way information about the photon is *redundantly encoded* in the environment.

Solution: The probability of measuring + on the nth probe is

$$\begin{aligned}
\Pr(+) &= \operatorname{Tr}\left(|\Psi\rangle\langle\Psi| I \otimes |+\rangle\langle+|_{p_n}\right) && (79)\\
&= \frac{1}{2}(|\langle P_0|+\rangle|^2 + |\langle P_1|+\rangle|^2) && (80)\\
&= \frac{1}{2}\left(\frac{(\cos\theta+\sin\theta)^2}{2} + \frac{(\cos\theta-\sin\theta)^2}{2}\right) && (81)\\
&= \frac{1}{2} && (82)
\end{aligned}$$

and therefore the prbability of measuring $-$ on the nth probe is $\Pr(-) = \frac{1}{2}$.

The state update rule tells us that for outcome +, the resultant state is

$$\begin{aligned}
|\Psi\rangle \to |\Phi_+\rangle &= \frac{1}{\sqrt{\Pr(+)}} I \otimes |+\rangle\langle+|_{p_n} |\Psi\rangle && (83)\\
&= \left(\langle+|P_0\rangle |0\rangle_S |P_0\rangle_{p_1} ... |P_0\rangle_{p_{n-1}} + e^{i\phi}\langle+|P_1\rangle |1\rangle_S |P_1\rangle_{p_1} ... |P_1\rangle_{p_{n-1}}\right) && (84)\\
&= \frac{1}{\sqrt{2}}\left((\cos\theta+\sin\theta) |0\rangle_S |P_0\rangle_{p_1} ... |P_0\rangle_{p_{n-1}} + e^{i\phi}(\cos\theta-\sin\theta) |1\rangle_S |P_1\rangle_{p_1} ... |P_1\rangle_{p_{n-1}}\right) |+\rangle_{p_n}
\end{aligned}$$

and similarly for outcome $-$ the resultant state is

$$|\Psi\rangle \to |\Phi_-\rangle = \frac{1}{\sqrt{2}}\left((\cos\theta-\sin\theta) |0\rangle_S |P_0\rangle_{p_1} ... |P_0\rangle_{p_{n-1}} + e^{i\phi}(\cos\theta+\sin\theta) |1\rangle_S |P_1\rangle_{p_1} ... |P_1\rangle_{p_{n-1}}\right) |-\rangle_{p_n}.$$

Now, before calculating the reduced density matrices on S, let's fix some notation to make it cleaner. Let $|0^k\rangle = |0\rangle_S |P_0\rangle_{p_1} ... |P_0\rangle_{p_{k-1}}$ and $|1^k\rangle = |1\rangle_S |P_1\rangle_{p_1} ... |P_1\rangle_{p_{k-1}}$. Then we can write

$$\begin{aligned}
|\Phi_+\rangle\langle\Phi_+| &= \frac{1}{2}((\cos\theta+\sin\theta)^2 |0^n\rangle\langle 0^n| + (\cos\theta+\sin\theta)(\cos\theta-\sin\theta)(e^{i\phi}|1^n\rangle\langle 0^n| + e^{-i\phi}|0^n\rangle\langle 1^n|) && (85)\\
&\quad +(\cos\theta-\sin\theta)^2|1^n\rangle\langle 1^n|)\otimes|+\rangle\langle +|_{p_n} && (86)\\
&= \frac{1}{2}((1+\sin(2\theta))|0^n\rangle\langle 0^n| + \cos(2\theta)(e^{i\phi}|1^n\rangle\langle 0^n| + e^{-i\phi}|0^n\rangle\langle 1^n|) && (87)\\
&\quad +(1-\sin(2\theta))|1^n\rangle\langle 1^n|)\otimes|+\rangle\langle +|_{p_n}. && (88)
\end{aligned}$$

Also, observe the following:

$$\begin{aligned}
\mathrm{Tr}_{p_{k-1}}\left(|0^k\rangle\langle 0^k|\right) &= |0^{k-1}\rangle\langle 0^{k-1}| && (89)\\
\mathrm{Tr}_{p_{k-1}}\left(|1^k\rangle\langle 1^k|\right) &= |1^{k-1}\rangle\langle 1^{k-1}| && (90)\\
\mathrm{Tr}_{p_{k-1}}\left(|1^k\rangle\langle 0^k|\right) &= \langle P_0|P_1\rangle |1^{k-1}\rangle\langle 0^{k-1}| && (91)\\
\mathrm{Tr}_{p_{k-1}}\left(|0^k\rangle\langle 1^k|\right) &= \langle P_0|P_1\rangle^* |0^{k-1}\rangle\langle 1^{k-1}|. && (92)
\end{aligned}$$

Now, to compute the reduced density matrix on S, we take the partial trace over systems p_1 through p_n iteratively, starting with p_n and decreasing:

$$\begin{aligned}
\rho_+ &= \mathrm{Tr}_{p_1...p_n}(|\Phi_+\rangle\langle\Phi_+|) && (93)\\
&= \mathrm{Tr}_{p_1...p_n}(\frac{1}{2}((1+\sin(2\theta))|0^n\rangle\langle 0^n| + \cos(2\theta)(e^{i\phi}|1^n\rangle\langle 0^n| + e^{-i\phi}|0^n\rangle\langle 1^n|) && (94)\\
&\quad +(1-\sin(2\theta))|1^n\rangle\langle 1^n|)\otimes|+\rangle\langle +|_{p_n}) && (95)\\
&= \mathrm{Tr}_{p_1...p_{n-1}}(\frac{1}{2}((1+\sin(2\theta))|0^n\rangle\langle 0^n| + \cos(2\theta)(e^{i\phi}|1^n\rangle\langle 0^n| + e^{-i\phi}|0^n\rangle\langle 1^n|) && (96)\\
&\quad +(1-\sin(2\theta))|1^n\rangle\langle 1^n|)) && (97)\\
&= \frac{1}{2}((1+\sin(2\theta))\mathrm{Tr}_{p_1...p_{n-1}}(|0^n\rangle\langle 0^n|) + \cos(2\theta)(e^{i\phi}\mathrm{Tr}_{p_1...p_{n-1}}(|1^n\rangle\langle 0^n|) + e^{-i\phi}\mathrm{Tr}_{p_1...p_{n-1}}(|0^n\rangle\langle 1^n|))\\
&\quad +(1-\sin(2\theta))\mathrm{Tr}_{p_1...p_{n-1}}(|1^n\rangle\langle 1^n|)) && (98)
\end{aligned}$$

Now, using our relations above, we can see that

$$\begin{aligned}
\mathrm{Tr}_{p_1...p_{n-1}}(|0^n\rangle\langle 0^n|) &= |0\rangle\langle 0|_S && (99)\\
\mathrm{Tr}_{p_1...p_{n-1}}(|1^n\rangle\langle 1^n|) &= |1\rangle\langle 1|_S && (100)\\
\mathrm{Tr}_{p_1...p_{n-1}}(|1^n\rangle\langle 0^n|) &= \langle P_0|P_1\rangle^{n-1}|1\rangle\langle 0|_S && (101)\\
\mathrm{Tr}_{p_1...p_{n-1}}(|0^n\rangle\langle 1^n|) &= \langle P_0|P_1\rangle^{*(n-1)}|0\rangle\langle 1|_S, && (102)
\end{aligned}$$

so we finally have that

$$\begin{aligned}
\rho_+ &= \frac{1}{2}((1+\sin(2\theta))|0\rangle\langle 0|_S + \cos(2\theta)\left(e^{i\phi}\langle P_0|P_1\rangle^{n-1}|1\rangle\langle 0|_S + e^{-i\phi}\langle P_0|P_1\rangle^{*(n-1)}|0\rangle\langle 1|_S\right)\\
&\quad +(1-\sin(2\theta))|1\rangle\langle 1|_S) && (103)\\
&= \frac{1}{2}\begin{pmatrix} 1+\sin(2\theta) & e^{-i\phi}\cos(2\theta)\langle P_0|P_1\rangle^{*(n-1)} \\ e^{i\phi}\cos(2\theta)\langle P_0|P_1\rangle^{n-1} & 1-\sin(2\theta)\end{pmatrix}. && (104)
\end{aligned}$$

Similarly, we find that for the $-$ outcome, the reduced state on the photon system is

$$\rho_- = \frac{1}{2}\begin{pmatrix} 1-\sin(2\theta) & e^{-i\phi}\cos(2\theta)\langle P_0|P_1\rangle^{*(n-1)} \\ e^{i\phi}\cos(2\theta)\langle P_0|P_1\rangle^{n-1} & 1+\sin(2\theta)\end{pmatrix}. \quad (105)$$

Now, the probability of success is $\Pr(+)\Pr(0|+) + \Pr(-)\Pr(1|-)$. To calculate this we need

$$\Pr(0|+) = \text{Tr}\,(\rho_+ |0\rangle\langle 0|) = \frac{1+\sin(2\theta)}{2} \tag{106}$$
$$\Pr(1|-) = \text{Tr}\,(\rho_- |1\rangle\langle 1|) = \frac{1+\sin(2\theta)}{2} \tag{107}$$

so we see that the probability of success is

$$\Pr(+)\Pr(0|+) + \Pr(-)\Pr(1|-) = \frac{1}{2}\frac{1+\sin(2\theta)}{2} + \frac{1}{2}\frac{1+\sin(2\theta)}{2} = \frac{1+\sin(2\theta)}{2}, \tag{108}$$

matching the results of problem 1c.

(f.) [2 Points] Now imagine that the experimenter measures the basis $\{|0\rangle, |1\rangle\}$ on the nth probe. What is the final state of the system after the measurement, for each of the two outcomes? Show that in this case, the fringe visibility of the interference pattern can be obtained from the expression for the fringe visibility in the case where the probe is ignored by simply replacing $\langle P_0|P_1\rangle$ with $\langle P_0|P_1\rangle^{n-1}$.

Solution: The probability of measuring 0 is

$$\Pr(0) = \text{Tr}\left(|\Psi\rangle\langle\Psi| I \otimes |0\rangle\langle 0|_{p_n}\right) \tag{109}$$
$$= \frac{1}{2}(|\langle P_0|0\rangle|^2 + |\langle P_1|0\rangle|^2) \tag{110}$$
$$= \cos^2\theta. \tag{111}$$

The computation of the reduced density matrices proceeds in an identical fashion to the previous part, except with different overlaps $\langle i|P_i\rangle$ instead of the $\langle \pm|P_i\rangle$ in the previous part. Replacing everything appropriately, one finds that the reduced density matrices for outcome 0 is

$$\rho_0 = \frac{1}{2}\begin{pmatrix} 1 & e^{-i\phi}\langle P_0|P_1\rangle^{*(n-1)} \\ e^{i\phi}\langle P_0|P_1\rangle^{n-1} & 1 \end{pmatrix}. \tag{112}$$

Since BS2 and the detectors together are equivalent to a $+/-$ measurement on this state, we find that

$$\Pr(D0) = \Pr(+) \tag{113}$$
$$= \text{Tr}\,(|+\rangle\langle +|\rho_0) \tag{114}$$
$$= \langle +|\rho_0|+\rangle \tag{115}$$
$$= \frac{1}{2}(\langle 0| + \langle 1|)\rho_0(|0\rangle + |1\rangle) \tag{116}$$
$$= \frac{1}{4}\left(1 + e^{i\phi}\langle P_0|P_1\rangle^{n-1} + e^{-i\phi}\langle P_0|P_1\rangle^{*(n-1)} + 1\right) \tag{117}$$
$$= \frac{1}{2}(1 + \langle P_0|P_1\rangle^{n-1}\cos\phi) \tag{118}$$

Similarly, for outcome 1, we have reduced state

$$\rho_1 = \frac{1}{2}\begin{pmatrix} 1 & -e^{-i\phi}\langle P_0|P_1\rangle^{*(n-1)} \\ -e^{i\phi}\langle P_0|P_1\rangle^{n-1} & 1 \end{pmatrix}. \tag{119}$$

which gives

$$\Pr(D0) \;=\; \frac{1}{2}(1 - \langle P_0|P_1\rangle^{n-1}\cos\phi). \tag{120}$$

In either case, the fringe visibility is now $V = \langle P_0|P_1\rangle^{n-1}$.

(g.) [1 Point] Suppose $\langle P_0|P_1\rangle = 1 - \epsilon$ where ϵ is much smaller than 1. In this case, how large must n be before the fringe visibility becomes negligible? Suppose $\langle P_0|P_1\rangle = \epsilon$. How large must n be before the fringe visibility becomes negligible?

Solution: The fringe visibility is $V = \langle P_0|P_1\rangle^{n-1}$. If $\langle P_0|P_1\rangle = 1 - \epsilon$, then

$$V \;=\; (1-\epsilon)^{n-1} = 1 - (n-1)\epsilon + \frac{1}{2}(n-1)(n-2)\epsilon^2 + O(\epsilon^3), \tag{121}$$

and we require $n = O(\frac{1}{\epsilon})$. For $n - 1 = \frac{N}{\epsilon}$, we have $V = (1-\epsilon)^{N/\epsilon} = e^{-N}$ as $\epsilon \to 0$. For $\langle P_0|P_1\rangle = \epsilon$, $V = \epsilon^{n-1}$, which is negligible for $n \geq 2$.

If one tries to define 'classical information' about a system as information about which there can be intersubjective agreement among many observers, then classical information is information that can be redundantly encoded in the different parts of the environment of the system. (This idea is called 'quantum Darwinism'.) The which-way information in our example was of this type. We have also seen that once the environment has achieved a redundant encoding of information about some basis of the system, then it is in practice infeasible for an experimenter who has access to only part of the environment to recover coherence for this basis.