Ph125c Spring 2017
Prof. Sean Carroll
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Homework - 6
Assigned TA: Charles Xu
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Due: 5:00pm, 5/24/2017

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1 Bohmian Trajectories [12 points]

Consider the Bohmian mechanics of a spinless nonrelativistic particle in one dimension, with mass m, position q, and potential $V(x)$, so that the Schrödinger and guidance equations are

$$i\frac{\partial}{\partial t}\Psi(x,t) = \left[-\frac{1}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\Psi(x,t) \tag{1}$$

$$\frac{dq}{dt} = \frac{1}{m}\mathrm{Im}\left(\frac{\nabla\Psi}{\Psi}\right)(q,t). \tag{2}$$

(a.) [2 Points] Show that the guidance equation can be written in the compact form

$$v = \frac{J}{|\Psi|^2}, \tag{3}$$

where $v = dq/dt$ is the particle velocity and J is a "current," an expression for which you will derive. (In more than one dimension the current would be a vector.)

(b.) [3 Points] Show that the wave function and current satisfy a continuity equation,

$$\frac{\partial}{\partial t}|\Psi(x,t)|^2 + \frac{\partial}{\partial x}J(x,t) = 0. \tag{4}$$

Argue (informally) that this implies that an initially equilibrium (distributed with respect to $|\Psi|^2$) ensemble of particles will remain in equilibrium as it evolves.

(c.) [4 Points] Define the "dwell time" of a particle, $\langle\tau_\Omega\rangle$, to be the expectation value of the amount of time the particle spends in a region $\Omega \subset \mathbb{R}^1$. Using Bohmian trajectories, show that this is given by

$$\langle\tau_\Omega\rangle = \int_{-\infty}^{\infty} dt \int_\Omega dx\, |\Psi(x,t)|^2. \tag{5}$$

(Hint: use the equilibrium condition for the distribution of Bohmian trajectories.) Use this to give a hand-waving argument that Bohmian mechanics should reproduce the usual interference phenomena of textbook quantum mechanics, such as the double-slit experiment.

(d.) [3 Points] Consider a one-dimensional problem, and choose some single trajectory, $Q(t)$, that a Bohmian particle could have, with a mass and potential as above. Define the probability that an actual particle is to the right of this fiducial trajectory by

$$P_R^{(Q)}(t) = \int_{Q(t)}^{\infty} dx\, |\Psi(x,t)|^2. \tag{6}$$

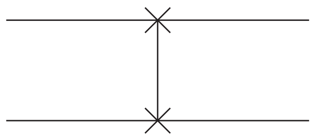
Show that this quantity is a constant over time. Argue that this implies (at least in one dimension) that Bohmian trajectories cannot cross each other.

2 Getting to Know Quantum Circuits [8 Points]

(a.) [2 Points] A SWAP gate takes an input state of two unentangled qubits and swaps them:

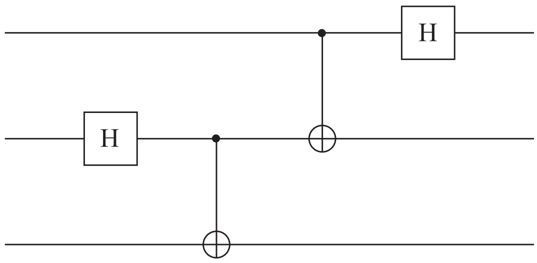
$$\text{SWAP} : |x\rangle \otimes |y\rangle \rightarrow |y\rangle \otimes |x\rangle. \tag{7}$$

It is generally portrayed thus:



Show how to construct a SWAP gate using only CNOT gates.

(b.) [2 Points] Consider the following quantum circuit, constructed from Hadamards and CNOTs:



Imagine we input an arbitrary qubit to the top register, and ancilla qubits $|0\rangle$ to the other two:

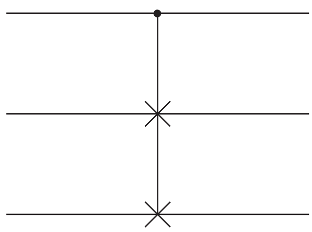
$$|\Psi_{\text{input}}\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes |0\rangle \otimes |0\rangle. \tag{8}$$

Derive the general form of the output state $|\Psi_{\text{output}}\rangle$, and calculate the probability for each possible outcome of measuring any of the final three qubits.

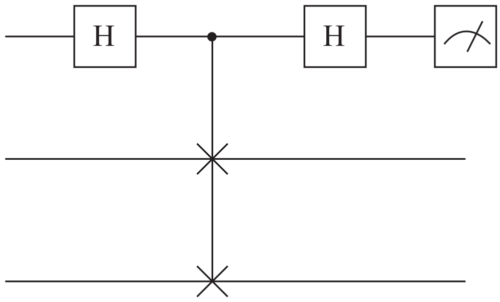
(c.) [2 Points] A *Fredkin gate*, also known as a CSWAP (controlled-SWAP) gate, maps 3-qubit states to 3-qubit states. Its action on basis states $|x_1x_2x_3\rangle$, where $x_i \in \{0, 1\}$, is the identity ($|x_1x_2x_3\rangle \rightarrow |x_1x_2x_3\rangle$) except for

$$|101\rangle \rightarrow |110\rangle \tag{9}$$
$$|110\rangle \rightarrow |101\rangle. \tag{10}$$

In other words, the second and third bits are swapped if the first is a 1, and left alone otherwise. It is generally portrayed thus:



Consider the following circuit, constructed from Hadamards and a Fredkin gate.



Imagine that we feed $|0\rangle$ into the first register, and two identical qubits $|\psi\rangle$ into the second and third:

$$|\Psi_{\text{input}}\rangle = |0\rangle \otimes |\psi\rangle \otimes |\psi\rangle. \tag{11}$$

What are the probabilities of getting 0 and 1 for the measurement outcomes on the first output qubit?

(d.) [2 Points] Same circuit as in part (d.), but now input two *orthogonal* states into the second and third registers:

$$|\Psi_{\text{input}}\rangle = |0\rangle \otimes |\psi\rangle \otimes |\phi\rangle \qquad \langle\psi|\phi\rangle = 0. \tag{12}$$

What are the probabilities for the measurement outcome of the first qubit now?