1 Bohmian Trajectories [12 points]

Consider the Bohmian mechanics of a spinless nonrelativistic particle in one dimension, with mass $m$, position $q$, and potential $V(x)$, so that the Schrödinger and guidance equations are

$$i \frac{\partial}{\partial t} \Psi(x, t) = \left[ -\frac{1}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right] \Psi(x, t)$$

(1)

$$\frac{dq}{dt} = \frac{1}{m} \text{Im} \left( \frac{\nabla \Psi}{\Psi} \right) (q, t).$$

(2)

(a.) [2 Points] Show that the guidance equation can be written in the compact form

$$v = \frac{J}{|\Psi|^2},$$

where $v = dq/dt$ is the particle velocity and $J$ is a “current,” an expression for which you will derive. (In more than one dimension the current would be a vector.)

(b.) [3 Points] Show that the wave function and current satisfy a continuity equation,

$$\frac{\partial}{\partial t} |\Psi(x, t)|^2 + \frac{\partial}{\partial x} J(x, t) = 0.$$  

(4)

Argue (informally) that this implies that an initially equilibrium (distributed with respect to $|\Psi|^2$) ensemble of particles will remain in equilibrium as it evolves.

(c.) [4 Points] Define the “dwell time” of a particle, $\langle \tau_\Omega \rangle$, to be the expectation value of the amount of time the particle spends in a region $\Omega \subset \mathbb{R}^1$. Using Bohmian trajectories, show that this is given by

$$\langle \tau_\Omega \rangle = \int_{-\infty}^{\infty} dt \int_\Omega dx |\Psi(x, t)|^2.$$  

(5)

(Hint: use the equilibrium condition for the distribution of Bohmian trajectories.) Use this to give a hand-waving argument that Bohmian mechanics should reproduce the usual interference phenomena of textbook quantum mechanics, such as the double-slit experiment.

(d.) [3 Points] Consider a one-dimensional problem, and choose some single trajectory, $Q(t)$, that a Bohmian particle could have, with a mass and potential as above. Define the probability that an actual particle is to the right of this fiducial trajectory by

$$P^Q_R(t) = \int_{Q(t)}^{\infty} dx |\Psi(x, t)|^2.$$  

(6)

Show that this quantity is a constant over time. Argue that this implies (at least in one dimension) that Bohmian trajectories cannot cross each other.
2 Getting to Know Quantum Circuits [8 Points]

(a.) [2 Points] A SWAP gate takes an input state of two unentangled qubits and swaps them:

\[ \text{SWAP} : |x\rangle \otimes |y\rangle \rightarrow |y\rangle \otimes |x\rangle. \]  

(7)

It is generally portrayed thus:

\[ \text{Diagram of SWAP gate} \]

Show how to construct a SWAP gate using only CNOT gates.

(b.) [2 Points] Consider the following quantum circuit, constructed from Hadamards and CNOTs:

\[ \text{Diagram of quantum circuit} \]

Imagine we input an arbitrary qubit to the top register, and ancilla qubits \(|0\rangle\) to the other two:

\[ |\Psi_{\text{input}}\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes |0\rangle \otimes |0\rangle. \]  

(8)

Derive the general form of the output state \(|\Psi_{\text{output}}\rangle\), and calculate the probability for each possible outcome of measuring any of the final three qubits.

(c.) [2 Points] A Fredkin gate, also known as a CSWAP (controlled-SWAP) gate, maps 3-qubit states to 3-qubit states. Its action on basis states \(|x_1x_2x_3\rangle\), where \(x_i \in \{0,1\}\), is the identity (\(|x_1x_2x_3\rangle \rightarrow |x_1x_2x_3\rangle\)) except for

\[ |101\rangle \rightarrow |110\rangle \]  

(9)

\[ |110\rangle \rightarrow |101\rangle. \]  

(10)

In other words, the second and third bits are swapped if the first is a 1, and left alone otherwise. It is generally portrayed thus:

\[ \text{Diagram of Fredkin gate} \]

Consider the following circuit, constructed from Hadamards and a Fredkin gate.
Imagine that we feed $|0\rangle$ into the first register, and two identical qubits $|\psi\rangle$ into the second and third:

$$|\Psi_{\text{input}}\rangle = |0\rangle \otimes |\psi\rangle \otimes |\psi\rangle.$$  \hfill (11)

What are the probabilities of getting 0 and 1 for the measurement outcomes on the first output qubit?

(d.) [2 Points] Same circuit as in part (d.), but now input two orthogonal states into the second and third registers:

$$|\Psi_{\text{input}}\rangle = |0\rangle \otimes |\psi\rangle \otimes |\phi\rangle \quad \langle \psi | \phi \rangle = 0.$$  \hfill (12)

What are the probabilities for the measurement outcome of the first qubit now?