Ph125c Spring 2017
Prof. Sean Carroll
seancarroll@gmail.com

Homework - 6 Solutions
Assigned TA: Charles Xu
cxu3@caltech.edu

---

1 Bohmian Trajectories [12 points]

Consider the Bohmian mechanics of a spinless nonrelativistic particle in one dimension, with mass m, position q, and potential $V(x)$, so that the Schrödinger and guidance equations are

$$i\frac{\partial}{\partial t}\Psi(x,t) = \left[-\frac{1}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\Psi(x,t) \tag{1}$$

$$\frac{dq}{dt} = \frac{1}{m}\text{Im}\left(\frac{\nabla\Psi}{\Psi}\right)(q,t). \tag{2}$$

(a.) [2 Points] Show that the guidance equation can be written in the compact form

$$v = \frac{J}{|\Psi|^2}, \tag{3}$$

where $v = dq/dt$ is the particle velocity and J is a "current," an expression for which you will derive. (In more than one dimension the current would be a vector.)

Solution: From eq. (2) we have

$$\begin{aligned} v &= \frac{1}{2im}\left(\frac{\nabla\Psi}{\Psi} - \frac{\nabla\Psi^*}{\Psi^*}\right) \\ &= \frac{1}{2im}\left(\frac{\Psi^*\nabla\Psi - \Psi\nabla\Psi^*}{|\Psi|^2}\right) \end{aligned}$$

from which it follows that

$$J = \frac{1}{2im}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \Psi\frac{\partial\Psi^*}{\partial x}\right).$$

(b.) [3 Points] Show that the wave function and current satisfy a continuity equation,

$$\frac{\partial}{\partial t}|\Psi(x,t)|^2 + \frac{\partial}{\partial x}J(x,t) = 0. \tag{4}$$

Argue (informally) that this implies that an initially equilibrium (distributed with respect to $|\Psi|^2$) ensemble of particles will remain in equilibrium as it evolves.

Solution: Using our expression for J and the product rule, we have

$$\begin{aligned} \frac{\partial J}{\partial x} &= \frac{1}{2im}\left(\frac{\partial\Psi^*}{\partial x}\frac{\partial\Psi}{\partial x} + \Psi^*\frac{\partial^2\Psi}{\partial x^2} - \frac{\partial\Psi}{\partial x}\frac{\partial\Psi^*}{\partial x} - \Psi\frac{\partial^2\Psi^*}{\partial x^2}\right) \\ &= \frac{1}{2im}\left(\Psi^*\frac{\partial^2\Psi}{\partial x^2} - \Psi\frac{\partial^2\Psi^*}{\partial x^2}\right). \end{aligned}$$

On the other hand, the Schrödinger equation eq. (1) gives us

$$\frac{\partial \Psi}{\partial t} = -\frac{1}{2mi}\frac{\partial^2 \Psi}{\partial x^2} - iV\Psi \quad \Rightarrow \quad \frac{\partial \Psi^*}{\partial t} = \frac{1}{2mi}\frac{\partial^2 \Psi^*}{\partial x^2} + iV\Psi^*$$

from which we obtain the other term,

$$\begin{aligned}
\frac{\partial |\Psi|^2}{\partial t} &= \Psi^* \frac{\partial \Psi}{\partial t} + \Psi \frac{\partial \Psi^*}{\partial t} \\
&= -\frac{1}{2mi}\Psi^* \frac{\partial^2 \Psi}{\partial x^2} - iV|\Psi|^2 + \frac{1}{2mi}\Psi\frac{\partial^2 \Psi^*}{\partial x^2} + iV|\Psi|^2 \\
&= \frac{1}{2im}\left(\Psi \frac{\partial^2 \Psi^*}{\partial x^2} - \Psi^* \frac{\partial^2 \Psi}{\partial x^2}\right) \\
&= -\frac{\partial J}{\partial x}
\end{aligned}$$

as desired. We may interpret this as follows: the time evolution $\frac{\partial |\Psi|^2}{\partial t}$ of the wavefunction is completely dictated by the Schrödinger equation, while eq. (4) describes how the ensemble of particle trajectories (with velocities given locally by $v = J|\Psi|^2$) responds to this. Any local increase or decrease in $|\Psi|^2$ is exactly compensated by a local outflow or inflow, respectively, of particle density given by $\frac{\partial J}{\partial x}$. Hence an ensemble of particles initially distributed according to $|\Psi|^2$ remains so as it evolves.

(c.) [4 Points] Define the "dwell time" of a particle, $\langle \tau_\Omega \rangle$, to be the expectation value of the amount of time the particle spends in a region $\Omega \subset \mathbb{R}^1$. Using Bohmian trajectories, show that this is given by

$$\langle \tau_\Omega \rangle = \int_{-\infty}^{\infty} dt \int_\Omega dx\, |\Psi(x,t)|^2. \tag{5}$$

(Hint: use the equilibrium condition for the distribution of Bohmian trajectories.) Use this to give a hand-waving argument that Bohmian mechanics should reproduce the usual interference phenomena of textbook quantum mechanics, such as the double-slit experiment.

Solution: For a single trajectory $q(t)$, we can define an exact dwell time

$$\tau_\Omega = \int_{-\infty}^{\infty} dt\, \mathbb{1}(q(t) \in \Omega)$$

in which we have introduced

$$\begin{aligned}
\mathbb{1}(q(t) \in \Omega) &= \begin{cases} 1 & \text{if } q(t) \in \Omega \\ 0 & \text{otherwise} \end{cases} \\
&= \int_\Omega dx\, \delta(x - q(t))
\end{aligned}$$

where $\delta(\cdot)$ is the Dirac delta. To obtain the expectation value $\langle \tau_\Omega \rangle$ we now perform an ensemble average over the equilibrium distribution of $q(t)$, which from part (b.) equals $|\Psi(q(t),t)|^2$:

$$\begin{aligned}\langle \tau_\Omega \rangle &= \left\langle \int_{-\infty}^{\infty} dt \int_\Omega dx\, \delta(x - q(t)) \right\rangle \\ &= \int_{-\infty}^{\infty} dt \int_\Omega dx\, \langle \delta(x - q(t)) \rangle \\ &= \int_{-\infty}^{\infty} dt \int_\Omega dx\, |\Psi(q(t),t)|^2 \delta(x - q(t)) \\ &= \int_{-\infty}^{\infty} dt \int_\Omega dx\, |\Psi(x,t)|^2\end{aligned}$$

as desired, where in the second line we used linearity of expectation. Thus in the Bohmian picture, the average *time* that a single particle in an equilibrium ensemble spends in a region Ω is proportional to the *probability* of its being found there according to conventional quantum mechanics. So the two formulations should yield statistically identical experimental outcomes.

(d.) [3 Points] Consider a one-dimensional problem, and choose some single trajectory, $Q(t)$, that a Bohmian particle could have, with a mass and potential as above. Define the probability that an actual particle is to the right of this fiducial trajectory by

$$P_R^{(Q)}(t) = \int_{Q(t)}^{\infty} dx\, |\Psi(x,t)|^2. \tag{6}$$

Show that this quantity is a constant over time. Argue that this implies (at least in one dimension) that Bohmian trajectories cannot cross each other.

Solution: Using the Leibniz integral rule we obtain

$$\begin{aligned}\frac{d}{dt} P_R^{(Q)}(t) &= -|\Psi(Q(t),t)|^2 \frac{dQ}{dt} + \int_{Q(t)}^{\infty} dx\, \frac{\partial}{\partial t} |\Psi(x,t)|^2 \\ &= -J(Q(t),t) - \int_{Q(t)}^{\infty} dx\, \frac{\partial}{\partial t} J(x,t) \\ &= -J(Q(t),t) - J(x,t)\Big|_{Q(t)}^{\infty} \\ &= -J(Q(t),t) + J(Q(t),t) \\ &= 0\end{aligned}$$

as desired. In line 2 we used the definition of J, eq. (3), for the first term and the continuity equation eq. (4) for the second; in line 4 we used the fact that $\lim_{x\to\infty} \Psi(x,t) = \lim_{x\to\infty} J(x,t) = 0$ for physical wavefunctions Ψ.

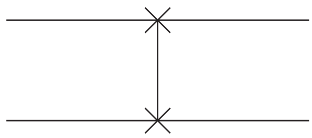
Now suppose for the sake of contradiction that two trajectories $Q(t)$ and $S(t)$ cross between times t_1 and t_2: that is to say, $Q(t_1) < S(t_1)$ but $Q(t_2) > S(t_2)$. Then it follows by continuity and the non-negativity of $|\Psi|^2$ that $P_R^{(Q)}(t_1) > P_R^{(S)}(t_1)$ while $P_R^{(Q)}(t_2) < P_R^{(S)}(t_2)$. But this contradicts the constancy of both $P_R^{(Q)}(t)$ and $P_R^{(S)}(t)$.

2 Getting to Know Quantum Circuits [8 Points]

(a.) [2 Points] A SWAP gate takes an input state of two unentangled qubits and swaps them:

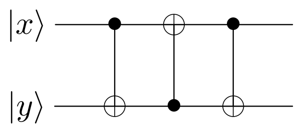
$$\text{SWAP} : |x\rangle \otimes |y\rangle \to |y\rangle \otimes |x\rangle. \tag{7}$$

It is generally portrayed thus:



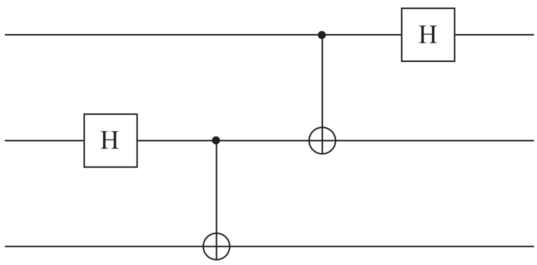
Show how to construct a SWAP gate using only CNOT gates.

Solution: The following circuit does the trick:



To see this, note that for $x, y \in 0, 1$ the input state $|x\rangle\,|y\rangle$ goes to $|x\rangle\,|y \oplus x\rangle$ after the first CNOT, then to $|x \oplus (y \oplus x)\rangle\,|y \oplus x\rangle = |y\rangle\,|y \oplus x\rangle$ after the second, and finally to $|y\rangle\,|(y \oplus x) \oplus y\rangle = |y\rangle\,|x\rangle$ after the third. Since this circuit is equivalent to the SWAP gate on a complete basis of inputs, this holds for arbitrary inputs.

(b.) [2 Points] Consider the following quantum circuit, constructed from Hadamards and CNOTs:



Imagine we input an arbitrary qubit to the top register, and ancilla qubits $|0\rangle$ to the other two:

$$|\Psi_{\text{input}}\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes |0\rangle \otimes |0\rangle. \tag{8}$$

Derive the general form of the output state $|\Psi_{\text{output}}\rangle$, and calculate the probability for each possible outcome of measuring any of the final three qubits.

Solution: The input state $|\Psi_{\text{input}}\rangle = \alpha\,|000\rangle + \beta\,|100\rangle$ evolves as follows.

After H(2): $\frac{\alpha}{\sqrt{2}}(|000\rangle + |010\rangle) + \frac{\beta}{\sqrt{2}}(|100\rangle + |110\rangle)$

After CNOT(2,3): $\frac{\alpha}{\sqrt{2}}(|000\rangle + |011\rangle) + \frac{\beta}{\sqrt{2}}(|100\rangle + |111\rangle)$

After CNOT(1,2): $\frac{\alpha}{\sqrt{2}}(|000\rangle + |011\rangle) + \frac{\beta}{\sqrt{2}}(|110\rangle + |101\rangle)$

After H(1): $|\Psi_{\text{output}}\rangle = \frac{\alpha}{2}(|000\rangle + |100\rangle + |011\rangle + |111\rangle) + \frac{\beta}{2}(|010\rangle - |110\rangle + |001\rangle - |101\rangle)$

Each of the three qubits takes the value 0 in two terms with amplitude $\frac{\alpha}{2}$ and another two terms with amplitude $\frac{\beta}{2}$; likewise for the value 1. Thus for each qubit,

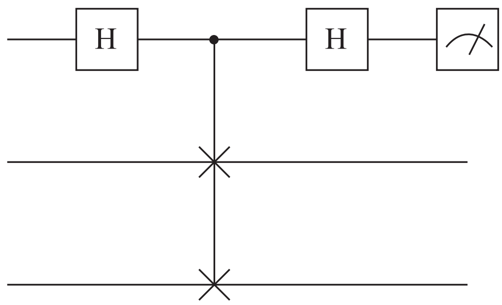
$$P(0) = P(1) = 2\left|\frac{\alpha}{2}\right|^2 + 2\left|\frac{\beta}{2}\right|^2 = \frac{|\alpha|^2 + |\beta|^2}{2} = \frac{1}{2}.$$

(c.) [2 Points] A *Fredkin gate*, also known as a CSWAP (controlled-SWAP) gate, maps 3-qubit states to 3-qubit states. Its action on basis states $|x_1x_2x_3\rangle$, where $x_i \in \{0,1\}$, is the identity ($|x_1x_2x_3\rangle \to |x_1x_2x_3\rangle$) except for

$$|101\rangle \to |110\rangle \tag{9}$$
$$|110\rangle \to |101\rangle. \tag{10}$$

In other words, the second and third bits are swapped if the first is a 1, and left alone otherwise.

Consider the following circuit, constructed from Hadamards and a Fredkin gate.



Imagine that we feed $|0\rangle$ into the first register, and two identical qubits $|\psi\rangle$ into the second and third:

$$|\Psi_{\text{input}}\rangle = |0\rangle \otimes |\psi\rangle \otimes |\psi\rangle. \tag{11}$$

What are the probabilities of getting 0 and 1 for the measurement outcomes on the first output qubit?

Solution: $|\Psi_{\text{input}}\rangle = |0\rangle|\psi\rangle|\psi\rangle$ evolves as follows:

After H(1): $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|\psi\rangle|\psi\rangle$

After CSWAP(1,2,3): $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|\psi\rangle|\psi\rangle$ since SWAP leaves the state $|\psi\rangle|\psi\rangle$ unchanged

After the second H(1): $|\Psi_{\text{output}}\rangle = |0\rangle|\psi\rangle|\psi\rangle = |\Psi_{\text{input}}\rangle$

It follows that for the first qubit $P(0) = 1$ and $P(1) = 0$.

(d.) [2 Points] Same circuit as in part (d.), but now input two *orthogonal* states into the second and third registers:

$$|\Psi_{\text{input}}\rangle = |0\rangle \otimes |\psi\rangle \otimes |\phi\rangle \qquad \langle\psi|\phi\rangle = 0. \tag{12}$$

What are the probabilities for the measurement outcome of the first qubit now?

Solution: $|\Psi_{\text{input}}\rangle = |0\rangle |\psi\rangle |\phi\rangle$ evolves as follows:

After H(1): $\frac{1}{\sqrt{2}} |0\rangle |\psi\rangle |\phi\rangle + \frac{1}{\sqrt{2}} |1\rangle |\psi\rangle |\phi\rangle$

After CSWAP(1,2,3): $\frac{1}{\sqrt{2}} |0\rangle |\psi\rangle |\phi\rangle + \frac{1}{\sqrt{2}} |1\rangle |\phi\rangle |\psi\rangle$

After the second H(1): $|\Psi_{\text{output}}\rangle = \frac{1}{2}(|0\rangle |\psi\rangle |\phi\rangle + |1\rangle |\psi\rangle |\phi\rangle + |0\rangle |\phi\rangle |\psi\rangle - |1\rangle |\phi\rangle |\psi\rangle)$

The four terms in this expression are all mutually orthogonal since $\langle 0|1\rangle = \langle \psi|\phi\rangle = 0$, and the first qubit appears twice as 0 and twice as 1, each with amplitude $\frac{1}{2}$. Thus $P(0) = P(1) = 2\left(\frac{1}{2}\right)^2 = \frac{1}{2}$.
