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1 More on the Bloch Sphere (10 points)

(a.) The state $|\Psi\rangle$ is parametrized on the Bloch Sphere by the angles (θ, ϕ) and hence, the state geometrically opposite to it will be characterized by (θ', ϕ'), where $\theta' = \pi - \theta$ and $\phi' = \pi + \phi$ if $0 \leq \phi < \pi$ or else $\phi' = \phi - \pi$ if $\pi \leq \phi < 2\pi$ to keep the Bloch angles in the range defined in the question. This gives us,

$$|\tilde{\Psi}\rangle = e^{i\gamma}\left(\cos\left(\frac{\pi-\theta}{2}\right)|0\rangle + e^{i(\phi+\pi)}\sin\left(\frac{\pi-\theta}{2}\right)|1\rangle\right) , \tag{1}$$

which gives us,

$$|\tilde{\Psi}\rangle = e^{i\gamma}\left(\sin\left(\frac{\theta}{2}\right)|0\rangle - e^{i\phi}\cos\left(\frac{\theta}{2}\right)|1\rangle\right) . \tag{2}$$

Now one can simply compute the inner product $\langle\tilde{\Psi}|\Psi\rangle$,

$$\langle\tilde{\Psi}|\Psi\rangle = \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) - \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) = 0 , \tag{3}$$

and hence states parametrized by opposite points on the Bloch Sphere are orthogonal.

(b.) Before we compute the unitary transformation of $\hat{\rho}$, let us first re-write the operator $\hat{U}$ in a more suggestive way. The $\hat{\mathbb{I}}$ term in the exponential of $\hat{U} = \exp\left(i\alpha\hat{\mathbb{I}} + i\beta\vec{u}\cdot\hat{\vec{\sigma}}\right)$ can be pulled out since it commutes with every other operator,

$$\hat{U} = \exp\left(i\alpha\hat{\mathbb{I}}\right)\,\exp\left(i\beta\vec{u}\cdot\hat{\vec{\sigma}}\right) . \tag{4}$$

We can now expand out the $\exp\left(i\beta\vec{u}\cdot\hat{\vec{\sigma}}\right)$ in powers of $\vec{u}\cdot\hat{\sigma}$,

$$\exp\left(i\beta\vec{u}\cdot\hat{\vec{\sigma}}\right) = \sum_{n=0}^{\infty}\frac{(i\beta)^n}{n!}\left(\vec{u}\cdot\hat{\vec{\sigma}}\right)^n . \tag{5}$$

Now using the operator product $(\vec{u}\cdot\hat{\vec{\sigma}})(\vec{v}\cdot\hat{\vec{\sigma}})$ we will compute in 1(c) below, notice, since $\vec{u}$ is a unit vector $|\vec{u}|^2 = 1$,

$$\left(\vec{u}\cdot\hat{\vec{\sigma}}\right)^n = \begin{cases} 1 & \text{for } n \text{ even} \\ \left(\vec{u}\cdot\hat{\vec{\sigma}}\right) & \text{for } n \text{ odd} \end{cases} \tag{6}$$

Thus, one can easily collect odd and even powers separately and they condense into trigonometric functions as follows,

$$\hat{U} = \exp\left(i\alpha\hat{\mathbb{I}}\right)\left(\cos(\beta)\hat{\mathbb{I}} + i\sin(\beta)\,\vec{u}\cdot\hat{\vec{\sigma}}\right) . \tag{7}$$

Since our goal is to prove that the Bloch vector $\vec{a}$ corresponding to $\hat{\rho} = \left(\hat{\mathbb{I}} + \vec{a}\cdot\hat{\vec{\sigma}}\right)/2$ is rotated by an angle 2β under the unitary transformation induced by $\hat{U}$, it will be convenient to divide up $\vec{a}$ into a component parallel to (the rotation axis set by $\hat{U}$)$\vec{u}$ and one perpendicular to it,

$$\vec{a} = (\vec{u}\cdot\vec{a})\,\vec{u} + (\vec{a} - (\vec{u}\cdot\vec{a})\,\vec{u}) \equiv \vec{a}_{||} + \vec{a}_{\perp} . \tag{8}$$

Now it's a matter of some algebra which I won't repeat here to avoid clutter. Be careful with all the i's, the signs and collect like terms to obtain the unitary transformation of $\hat{\rho}$ as,

$$\hat{U}\hat{\rho}\hat{U}^\dagger = \frac{1}{2}\left[\hat{\mathbb{I}} + (\vec{u}\cdot\vec{a})\,\vec{u}\cdot\hat{\vec{\sigma}} + \cos(2\beta)\vec{a}_\perp\cdot\hat{\vec{\sigma}} - \sin(2\beta)\,(\vec{u}\times\vec{a}_\perp)\cdot\hat{\vec{\sigma}}\right] . \tag{9}$$

This can be interpreted rather straightforwardly. The Bloch vector corresponding to this density operator is,

$$\vec{a}' = (\vec{u}\cdot\vec{a})\,\vec{u} + \cos(2\beta)\vec{a}_\perp - \sin(2\beta)\,(\vec{u}\times\vec{a}_\perp) , \tag{10}$$

where one can easily see that the component of $\vec{a}$ parallel to $\vec{u}$ did not rotate while the perpendicular component $\vec{a}_\perp$ underwent a rotation of 2β around the axis $\vec{u}$. This rotation has a left-handed sense as can be seen by the $(-)$ sign with the $(\vec{u}\times\vec{a}_\perp)$ term and this can be traced to our construction of the unitary operator $\hat{U} = \exp\left(i\alpha\hat{\mathbb{I}}\right)\,\exp\left(i\beta\vec{u}\cdot\hat{\vec{\sigma}}\right)$ and the transformation being affected via $\hat{U}\hat{\rho}\hat{U}^\dagger$. With the same $\hat{U}$, the transformation $\hat{U}^\dagger\hat{\rho}\hat{U}$ would result in a right-handed rotation of $\vec{a}$ by angle 2β about $\vec{u}$.

While I have not explicitly done the $\hat{U}\hat{\rho}\hat{U}^\dagger$ algebra here, if you are having any trouble with it, please feel free to get it touch with me.

(c.) Let us expand out the required operator product,

$$(\vec{u}\cdot\hat{\vec{\sigma}})(\vec{v}\cdot\hat{\vec{\sigma}}) = \sum_{j=1}^{3}\sum_{k=1}^{3} u_j v_k \hat{\sigma}_j\hat{\sigma}_k . \tag{11}$$

Now, we wish to harness the algebra of the Pauli operators to simplify this. In particular, write,

$$\hat{\sigma}_j\hat{\sigma}_k = \frac{1}{2}\left[\hat{\sigma}_j,\hat{\sigma}_k\right] + \frac{1}{2}\left\{\hat{\sigma}_j,\hat{\sigma}_k\right\} = \frac{1}{2}\sum_{l=1}^{3} 2i\epsilon_{jkl}\hat{\sigma}_l + \frac{1}{2}2\delta_{jk}\hat{\mathbb{I}} . \tag{12}$$

Substituting this back in the operator product we get,

$$(\vec{u}\cdot\hat{\vec{\sigma}})(\vec{v}\cdot\hat{\vec{\sigma}}) = \sum_{j=1}^{3}\sum_{k=1}^{3} u_j v_k\left(\delta_{jk}\hat{\mathbb{I}} + \sum_{l=1}^{3} i\epsilon_{jkl}\hat{\sigma}_l\right) . \tag{13}$$

Using the antisymmetry of the Levi-Cevita symbol twice(or cyclicity),

$$(\vec{u}\cdot\hat{\vec{\sigma}})(\vec{v}\cdot\hat{\vec{\sigma}}) = \sum_{j=1}^{3}\sum_{k=1}^{3} u_j v_k\left(\delta_{jk}\hat{\mathbb{I}} + \sum_{l=1}^{3} i\epsilon_{ljk}\hat{\sigma}_l\right) . \tag{14}$$

Performing this sum and noticing the vector expressions $\vec{u}\cdot\vec{v} = \sum_{j=1}^{3} u_j v_j$ and $(\vec{u}\times\vec{v})_l = i\sum_{j,k=1}^{3}\epsilon_{ljk}u_j v_k$, we obtain,

$$(\vec{u}\cdot\hat{\vec{\sigma}})(\vec{v}\cdot\hat{\vec{\sigma}}) = (\vec{u}\cdot\vec{v})\hat{\mathbb{I}} + (\vec{u}\times\vec{v})\cdot\hat{\vec{\sigma}} . \tag{15}$$

Now we square the expression for $\hat{\rho}$,

$$\hat{\rho}^2 = \frac{1}{2}\left(\hat{\mathbb{I}} + \vec{a}\cdot\hat{\vec{\sigma}}\right)\frac{1}{2}\left(\hat{\mathbb{I}} + \vec{a}\cdot\hat{\vec{\sigma}}\right) = \frac{1}{4}\left(\hat{\mathbb{I}} + 2\vec{a}\cdot\hat{\vec{\sigma}} + (\vec{a}\cdot\hat{\vec{\sigma}})(\vec{a}\cdot\hat{\vec{\sigma}})\right) . \tag{16}$$

We use the operator product just computed for the special case of $\vec{u} = \vec{v} = \vec{a}$ for which it reads $(\vec{a}\cdot\hat{\vec{\sigma}})(\vec{a}\cdot\hat{\vec{\sigma}}) = |\vec{a}|^2\hat{\mathbb{I}}$ to give,

$$\hat{\rho}^2 = \frac{1}{4}\left((1+|\vec{a}|^2)\hat{\mathbb{I}} + 2\vec{a}\cdot\hat{\vec{\sigma}}\right) . \tag{17}$$

2 Purity! (5 points)

(a.) The density operator of a pure state $|\Psi\rangle$ can be written as $\hat{\rho}_{\text{pure}} = |\Psi\rangle\langle\Psi|$ and by construction, we consider normalized states obeying $\langle\Psi|\Psi\rangle = 1$ which will ensure our density operator has unit trace. Now we simply square $\hat{\rho}_{\text{pure}}$ to obtain,

$$\hat{\rho}^2_{\text{pure}} = (|\Psi\rangle\langle\Psi|)(|\Psi\rangle\langle\Psi|) = |\Psi\rangle(\langle\Psi|\Psi\rangle)\langle\Psi| = |\Psi\rangle\langle\Psi| \,, \tag{18}$$

and hence $\hat{\rho}_{\text{pure}}$ is idempotent. The purity of such a pure state is,

$$\text{Purity}(\hat{\rho}_{\text{pure}}) = \text{Tr}\,(\hat{\rho}^2_{\text{pure}}) = \text{Tr}\,(\hat{\rho}_{\text{pure}}) = 1 \,. \tag{19}$$

(b.) One can argue/calculate what the most mixed state looks like in various ways. Consider $\{\lambda_j\,,\ j = 1, 2, \cdots, d\}$ be the set of eigenvalues of a general (possibly non-pure)$\hat{\rho}$. Positivity of $\hat{\rho}$ ensures $\lambda_j \geq 0\ \forall\ j$. The unit trace condition on $\hat{\rho}$ implies $\sum_{j=1}^{d} \lambda_j = 1$ and the purity is simply,

$$\text{Purity}(\hat{\rho}) = \sum_{j=1}^{d} \lambda_j^2 \,, \tag{20}$$

since one can always diagonalize the corresponding density matrix and have its eigenvalues on the diagonal,

$$\hat{\rho} = \sum_{j=1}^{d} \lambda_j \,|\lambda_j\rangle\langle\lambda_j| \,, \tag{21}$$

where $\{|\lambda_j\rangle\}$ is the set of corresponding (complete) eigenvectors. Now the question reduces to minimizing the Purity subject to the unit trace constraint. One can solve this via Lagrange multipliers, but I will prove it using the Cauchy-Schwarz inequality,

$$\left|\sum_{j=1}^{d} \lambda_j\right|^2 \leq \left(\sum_{j=1}^{d} \lambda_j^2\right)\left(\sum_{j=1}^{d} (1)^2\right) \,. \tag{22}$$

Now the left side of this equation is simply one due to the unit trace condition and hence, we obtain,

$$\text{Purity}(\hat{\rho}) = \sum_{j=1}^{d} \lambda_j^2 \geq \frac{1}{d} \,. \tag{23}$$

The maximally mixed state has minimum purity given by $\text{Purity}(\hat{\rho}_{\text{max mix}}) = 1/d$, corresponding to $\lambda_j = 1/d\ \forall\ j$ and is given by,

$$\text{Purity}(\hat{\rho}_{\text{max mix}}) = \frac{1}{d}\hat{\mathbb{I}} \,. \tag{24}$$

(c.) Now back to the qubit case from Problem 1. Recall the expression for $\hat{\rho}^2$ from Eq. (17),

$$\hat{\rho}^2 = \frac{1}{4}\left((1 + |\vec{a}|^2)\hat{\mathbb{I}} + 2\vec{a}\cdot\hat{\vec{\sigma}}\right) \,. \tag{25}$$

Now, the purity of this state is simply,

$$\text{Purity}(\hat{\rho}) = \text{Tr}\,(\hat{\rho}^2) = \frac{1}{4}(1 + |\vec{a}|^2) \times 2 = \frac{1}{2}(1 + |\vec{a}|^2)\,, \tag{26}$$

since $\text{Tr}\,(\hat{\mathbb{I}}) = 2$ for a qubit and $\text{Tr}\,(\hat{\sigma}_j) = 0\;,\; j = 1, 2, 3$. Now since the purity of a pure state is unity, it is easily seen that for a pure qubit state $|\vec{a}|^2 = |\vec{a}| = 1$ and hence such pure states lie on the surface of the Bloch Sphere, while mixed states lie within it.

3 One Entangled Evening...spent doing Homework (10 points)

(a.) The first thing to notice here is that the state $|\Psi\rangle$ given in the Schmidt decomposition for $\mathcal{H} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$ with $\dim \mathcal{A} = \dim \mathcal{B} = d$,

$$|\Psi\rangle = \sum_n a_n \, |\lambda_n\rangle_A \otimes |\phi_n\rangle_B \;, \tag{27}$$

has expansion coefficients $\{a_n, n = 1, 2, \cdots, d\}$ which are squares of the eigenvalues of the reduced density matrices $\rho_A = \text{Tr}\,_B(\rho)$ and $\rho_B = \text{Tr}\,_A(\rho)$ *i.e.* $\lambda_n = a_n^2$. If you are not sure about this, please refer to your class notes. Positivity and unit trace conditions on the density operator translate into $\lambda_n \geq 0 \;\forall\; n = 1, 2, \cdots, d$ and $\sum_{n=1}^{d} \lambda_n = 1$. Let us define a new quantity, call it β, defined as the inverse of our target function we wish to optimize,

$$\beta = \frac{1}{\kappa} = \sum_{n=1}^{d} \lambda_n^2 \,. \tag{28}$$

The maximum value of β which corresponds to minimum value of κ occurs when we have only one non-zero eigenvalue *i.e.* $\lambda_j = 1$ for some j and $\lambda_{k \neq j} = 0$ which corresponds to $\kappa = 1$. Since only one of the coefficients $a_j = 1$ is non-zero, we have a product state which is an unentangled state. On the other hand, to maximize κ, we minimize β. This is on exactly the same lines we followed while minimizing purity in Problem (2). Now under these conditions, it is easily seen that β is minimized for $\lambda_n = 1/d \;\forall\; n$ which corresponds to a maximum value of $\kappa = d =$ dimension of the (lower, in this case of either) Hilbert space. One can use a simple Lagrange multiplier constraint optimization to show this too where the condition is simply the unit trace condition $\sum_{n=1}^{d} \lambda_n = 1$.

(b.) The Bell states, for $\mathcal{H} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$ with $\dim \mathcal{A} = \dim \mathcal{B} = 2$ written in the standard $\{|0\rangle\,, |1\rangle\}$ basis are given by,

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\,(|00\rangle + |11\rangle) \tag{29}$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}\,(|00\rangle - |11\rangle) \tag{30}$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}\,(|01\rangle + |10\rangle) \tag{31}$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\,(|01\rangle - |10\rangle) \;. \tag{32}$$

While it is evident from their structure that the Bell states are already Schmidt decomposed, one can also check this by computing the reduced density matrices $\rho_A = \text{Tr}\,_B(\rho)$ and $\rho_B = \text{Tr}\,_A(\rho)$. To illustrate this, consider the Bell state $|\Phi^+\rangle$ having the density operator $\hat{\rho}_{\Phi^+}$,

$$\hat{\rho}_{\Phi^+} = \frac{1}{2}(\,|00\rangle\,\langle 00| + |11\rangle\,\langle 11| + |11\rangle\,\langle 00| + |00\rangle\,\langle 11|\,)\,. \tag{33}$$

The corresponding reduced density operators can be easily computed,

$$\rho_A = \mathrm{Tr}_B(\rho) = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \,, \tag{34}$$

$$\rho_B = \mathrm{Tr}_A(\rho) = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \,, \tag{35}$$

in their respective operator spaces. Now one can recognize $\lambda_1 = \lambda_2 = 1/2$ and hence $\kappa = 2 = \dim\mathcal{A} = \dim\mathcal{B}$, hence from the result of (a), we have proved that this state is maximally entangled. Similar analysis holds for the other Bell states which are also maximally entangled.

(c.) Onto the Werner state now. The density operator of the Werner state is,

$$\hat{\rho} = |W\rangle\langle W| = |\alpha|^2 |001\rangle\langle 001| + |\beta|^2 |010\rangle\langle 010| + |\gamma|^2 |100\rangle\langle 100| + \alpha\beta^* |001\rangle\langle 010| + \tag{36}$$

$$\alpha\gamma^* |001\rangle\langle 100| + \beta\alpha^* |010\rangle\langle 001| + \beta\gamma^* |010\rangle\langle 100| + \gamma\alpha^* |100\rangle\langle 001| + \gamma\beta^* |100\rangle\langle 010| \,. \tag{37}$$

One can now trace over qubits $\mathcal{B}$ and $\mathcal{C}$ to get the reduced density operator for $\mathcal{A}$. Notice, since we are taking the trace over the last two qubits, only terms with identical entires on the ket and its dual for each $|.\rangle\langle .|$ survive,

$$\hat{\rho}_A = \mathrm{Tr}_{BC}(\hat{\rho}) = \left(|\alpha|^2 + |\beta|^2\right) |0\rangle\langle 0| + |\gamma|^2 |1\rangle\langle 1| \,, \tag{38}$$

and one can easily see that the corresponding density matrix in the standard basis is diagonal with eigenvalues $\left(|\alpha|^2 + |\beta|^2\right)$ and $|\gamma|^2$, which gives the von Neumann entropy of qubit $\mathcal{A}$ to be,

$$S(\hat{\rho}_A) = -\mathrm{Tr}\,(\hat{\rho}_A \log \hat{\rho}_A) = -\left(|\alpha|^2 + |\beta|^2\right)\log\left(|\alpha|^2 + |\beta|^2\right) - |\gamma|^2 \log\left(|\gamma|^2\right) \,. \tag{39}$$

Since none of α, β or γ are zeros, this is well defined.