Ph125c Spring 2017
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Homework - 2 Solutions
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1 Traces, Traces Everywhere (5 points)

(a.) Okay, so the time evolved state is given by,

$$\hat{\rho}(t) = \hat{U}(t)\hat{\rho}(0)\hat{U}^\dagger(t) \; , \tag{1}$$

from which we can compute $\hat{\rho}^n(t)$ using the fact that $\hat{U}$ is unitary *i.e.* $\hat{U}(t)\hat{U}^\dagger(t) = \hat{U}^\dagger(t)\hat{U}(t) = \hat{\mathbb{I}}$,

$$\hat{\rho}^n(t) = \hat{U}(t)\hat{\rho}^n(0)\hat{U}^\dagger(t) \; . \tag{2}$$

Now we trace over this and use the cyclic property of trace $\mathrm{Tr}\,(ABC) = \mathrm{Tr}\,(CAB) = \mathrm{Tr}\,(BCA)$ to get,

$$\mathrm{Tr}\;(\hat{\rho}^n(t)) = \mathrm{Tr}\;(\hat{\rho}^n(0)) = \sum_i \lambda_i^n \; , \tag{3}$$

where $\{\lambda_i\}$ is the set of eigenvalues of $\hat{\rho}(0)$.

(b.) While one can compute the asked traces and determinant explicitly by considering a generic qubit state and its corresponding 2×2 density operator, we will follow a much compact approach, harnessing properties of traces. In what follows, denote λ_1 and λ_2 to be the eigenvalues of the density matrix of the qubit state $\hat{\rho}$.
It is easily seen that,

$$\mathrm{Tr}\,(\hat{\rho}^n) = \lambda_1^n + \lambda_2^n \; . \tag{4}$$

In particular we have $\mathrm{Tr}\,(\hat{\rho}) = \lambda_1 + \lambda_2 = 1$ since the density operator has unit trace and $\mathrm{Tr}\,(\hat{\rho}^2) = \lambda_1^2 + \lambda_2^2$. Also, one can notice that $\det(\rho) = \lambda_1\lambda_2$. Okay, now let's use some kindergarden arithmetic: $(\lambda_1+\lambda_2)^2 = \lambda_1^2+\lambda_2^2+2\lambda_1\lambda_2$, which in our context is gives us the determinant we are looking for,

$$\det(\rho) = \frac{1}{2}\left(\mathrm{Tr}^{\,2}(\hat{\rho}) - \mathrm{Tr}\,(\hat{\rho}^2)\right) = \frac{1}{2}\left(\mathrm{Tr}\,(\hat{\rho}) - \mathrm{Tr}\,(\hat{\rho}^2)\right) \; . \tag{5}$$

Now, consider $\lambda_1^3 + \lambda_2^3 = (\lambda_1 + \lambda_2)(\lambda_1^2 + \lambda_2^2 - \lambda_1\lambda_2)$, from which one can directly write,

$$\mathrm{Tr}\,(\hat{\rho}^3) = \mathrm{Tr}\,(\rho)\left(\mathrm{Tr}\,(\hat{\rho}^2) - \det(\rho)\right) = \mathrm{Tr}\,(\rho)\left(\frac{3}{2}\mathrm{Tr}\,(\hat{\rho}^2) - \frac{1}{2}\mathrm{Tr}\,(\hat{\rho})\right) \tag{6}$$

Of course, $\mathrm{Tr}\,\hat{\rho} = 1$ everywhere above. If you feel like, you can also compute these by considering a generic qubit state $\hat{\rho} = \hat{\mathbb{I}}/2 + \vec{a}\cdot\vec{\sigma}/2$ and find $\hat{\rho}^2$ and $\hat{\rho}^3$ and find the traces and determinant asked for.

2 More on von Neumann Entropy (5 points)

(a.) This one can be argued simply by observing the fact that a unitary transformation of $\hat{\rho}$ simply corresponds to a global basis change and eigenvalues of a matrix are left invariant under a basis change/unitary transformation and hence, the von Neumann entropy, which can be constructed using only the eigenvalues of $\hat{\rho}$ only is invariant too.

(b.) Onto proving concavity now. First, it is important to mention that, in general, $\hat{\rho}$, $\hat{\rho}_1$ and $\hat{\rho}_2$ will not commute and hence will *not* be simultaneously diagonalizable. Let us work in the eigenbasis of $\hat{\rho}$ which we label $|\rho_m\rangle$ with corresponding eigenvalue ρ_m (We will use single subscripts for eigen properties and double indices will be used to denote matrix elements.) The von Neumann entropy of $\hat{\rho}$ is simply given by,

$$S(\hat{\rho}) = \sum_m -\rho_m \log(\rho_m) \equiv \sum_m s(\rho_m) \,, \tag{7}$$

where the function $s(x) = -x\log(x)$. This is a concave in the domain $[0, 1]$ since its second derivative $f'' \leq 0$ and hence it satisfies a concavity condition, for any $\alpha \in [0, 1]$

$$s((1-\alpha)x + \alpha y) \geq (1-\alpha)s(x) + \alpha s(y) \,. \tag{8}$$

We will use this property heavily in this proof. Notice how the eigenvalue ρ_m of $\hat{\rho}$ can be written as,

$$\rho_m = \langle\rho_m|\hat{\rho}|\rho_m\rangle = p_1 \langle\rho_m|\hat{\rho}_1|\rho_m\rangle + p_2 \langle\rho_m|\hat{\rho}_2|\rho_m\rangle \,, \tag{9}$$

where we used $\hat{\rho} = p_1\hat{\rho}_1 + p_2\hat{\rho}_2$ as given in the question. Now since each of the terms in the summand of Eq. (7) is concave, we can use the concavity property of Eq. (8) to write,

$$s(\rho_m) \geq p_1 s(\langle\rho_m|\hat{\rho}_1|\rho_m\rangle) + p_2 s(\langle\rho_m|\hat{\rho}_2|\rho_m\rangle) \,, \tag{10}$$

and a simple sum over m reads,

$$S(\hat{\rho}) \geq p_1 \sum_m s(\langle\rho_m|\hat{\rho}_1|\rho_m\rangle) + p_2 \sum_m s(\langle\rho_m|\hat{\rho}_2|\rho_m\rangle) \,. \tag{11}$$

Now life would have been nice and simple had $\hat{\rho}_1$ and $\hat{\rho}_2$ been diagonal in the $|\rho_m\rangle$ (eigenbasis of $\hat{\rho}$) basis and we could have identified the sums on the right side of Eq. (11) as the von Neumann entropies $S(\hat{\rho}_1)$ and $S(\hat{\rho}_2)$, but well, in general $s(\langle\rho_m|\hat{\rho}_1|\rho_m\rangle)$ and $s(\langle\rho_m|\hat{\rho}_1|\rho_m\rangle)$ are *not* the eigenvalues of $\hat{\rho}_1$ and $\hat{\rho}_2$, but just the diagonal entries of the density matrices of $\hat{\rho}_1$ and $\hat{\rho}_2$ in the $\{|\rho_m\rangle\}$ basis. Let us now try and make contact with the von Neumann entropies of $\hat{\rho}_1$ and $\hat{\rho}_2$ using these diagonal entries.

Let me state and prove a general result that we will end up needing. We will now show that,

$$-\sum_n \lambda_{nn} \log(\lambda_{nn}) \geq S(\hat{\rho}) \,, \tag{12}$$

where $\hat{\rho}$ is any density operator and λ_{nn} are its diagonal entries in *any* basis, not necessarily its eigenbasis (The inequality will become an equality when we work in the eigenbasis). Let us write $\hat{\rho}$ in both its eigenbasis $\{|\rho_m\rangle\}$ and some other basis $\{|\lambda_n\rangle\}$,

$$\hat{\rho} = \sum_m \rho_m |\rho_m\rangle\langle\rho_m| = \sum_{n,n'} \lambda_{nn'} |\lambda_{n'}\rangle\langle\lambda_n| \,. \tag{13}$$

To connect these two and be able to write the diagonal entries λ_{nn} in terms of the eigenvalues ρ_m, let's introduce two complete set of states as follows,

$$\hat{\rho} = \sum_m \rho_m \sum_{n,n'} |\lambda_n\rangle\langle\lambda_n| \left(|\rho_m\rangle\langle\rho_m|\right) |\lambda_{n'}\rangle\langle\lambda_{n'}| \,, \tag{14}$$

from which we can read the diagonal entries λ_{nn} simply,

$$\lambda_{nn} = \sum_m \rho_m \left|\langle\lambda_n|\rho_m\rangle\right|^2 \ . \tag{15}$$

These overlaps are expansion coefficients of eigenstates $|\rho_m\rangle$ in the $\{|\lambda_n\rangle\}$ basis and one can easily they verify they satisfy,

$$\sum_n \left|\langle\lambda_n|\rho_m\rangle\right|^2 = 1 \ . \tag{16}$$

Hence, we notice, each diagonal term is a weighted sum of eigenvalues as shown by Eq. (15) with all weights summing to unity and hence we can invoke concavity again to write,

$$-\lambda_{nn}\log\left(\lambda_{nn}\right) \geq \sum_m \left|\langle\lambda_n|\rho_m\rangle\right|^2 \left(-\rho_m \log\left(\rho_m\right)\right) \tag{17}$$

This concludes our digression proof. Return to where we left off for concavity and now we simply apply this general inequality result to each of the $\hat{\rho}_1$ and $\hat{\rho}_2$ diagonal terms of Eq. (11) and perform the sum to get the desired concavity answer,

$$S(\hat{\rho}) \geq p_1 S(\hat{\rho}_1) + p_2 S(\hat{\rho}_2) \ . \tag{18}$$

(c.) Consider the separable state $\hat{\rho}_{AB} = \hat{\rho}_A \otimes \hat{\rho}_B$. For concreteness take $\dim(A) = d_A$ and $\dim(B) = d_B$ and hence the dimension of Hilbert space on which $\hat{\rho}_{AB}$ lives is simply $d = d_A d_B$. Let's label the d eigenvalues of $\hat{\rho}_{AB}$ as λ_k with $k = 1, 2, \cdots, d$. Also, label the eigenvalues of $\hat{\rho}_A$ as α_i, $i = 1, 2, \cdots, d_A$ and those of $\hat{\rho}_B$ as β_j, $j = 1, 2, \cdots, d_B$. One can now construct a bijective map which relates the $d = d_A d_B$ eigenvalues of $\hat{\rho}_{AB}$ with those of $\hat{\rho}_A$ and $\hat{\rho}_B$,

$$\lambda_k = \alpha_i \beta_j \ , \ \ k \equiv (i,j) \text{ with } i = 1, 2, \cdots, d_A \text{ and } k = 1, 2, \cdots, d_B \ . \tag{19}$$

Now, let's get to the entropy.

$$S(\hat{\rho}_{AB}) = \sum_{k=1}^{d} \lambda_k \log\left(\lambda_k\right) = \sum_{i=1}^{d_A}\sum_{j=1}^{d_B} (\alpha_i\beta_j)\log\left(\alpha_i\beta_j\right) . \tag{20}$$

We can now split the log to get,

$$S(\hat{\rho}_{AB}) = \sum_{j=1}^{d_B} \beta_j \sum_{i=1}^{d_A} \alpha_i \log\left(\alpha_i\right) + \sum_{i=1}^{d_A} \alpha_i \sum_{j=1}^{d_B} \beta_j \log\left(\beta_j\right) . \tag{21}$$

Now notice, $S(\hat{\rho}_A) = \sum_{i=1}^{d_A} \alpha_i \log\left(\alpha_i\right)$ and $S(\hat{\rho}_B) = \sum_{j=1}^{d_B} \beta_j \log\left(\beta_j\right)$ and also that the sum of eigenvalues of any density operator is its trace which is unity, *i.e.* $\sum_{i=1}^{d_A} \alpha_i = \sum_{j=1}^{d_B} \beta_j = 1$, which gives us the required result,

$$S(\hat{\rho}_{AB}) = S(\hat{\rho}_A) + S(\hat{\rho}_B) \ . \tag{22}$$

3 Let's measure a GHZ! (5 points)

(a.) Let us construct the projection operators corresponding to measurement in the $\{|\pm x\rangle\}$ basis,

$$\hat{P}_+ = |+x\rangle\langle +x| \ , \tag{23}$$

$$\hat{P}_- = |-x\rangle\langle -x| \ . \tag{24}$$

One can now easily see that this set of projection operators satisfies all the required conditions for being a *Projective Value Measurement (PVM)*. The set of projection operators are

1. Hermitian: $\hat{P}_+^\dagger = \hat{P}_+$, $\hat{P}_-^\dagger = \hat{P}_-$,
2. Complete: $\hat{P}_+ + \hat{P}_- = |0\rangle\langle 0| + |1\rangle\langle 1| = \hat{\mathbb{I}}$,
3. Orthogonal Projectors: $\hat{P}_+\hat{P}_+ = \hat{P}_+$, $\hat{P}_-\hat{P}_- = \hat{P}_-$, $\hat{P}_+\hat{P}_- = \hat{P}_-\hat{P}_+ = 0$
4. Positive: eigenvalues of $\hat{P}_+$ and $\hat{P}_-$ are 0 and 2 which are ≥ 0

(b.) The pre-measurement GHZ density operator is simply,

$$\hat{\rho}_0 = |\text{GHZ}\rangle\langle\text{GHZ}| = \frac{1}{2}\left(|000\rangle\langle 000| + |000\rangle\langle 111| + |111\rangle\langle 000| + |111\rangle\langle 111|\right) \ , \tag{25}$$

where the convention I am following labels subsystems in states and their duals as $|ABC\rangle$ and $\langle ABC|$, respectively.

Okay, let's compute some projections now, but first notice the following overlaps which will be useful,

$$\langle +x|0\rangle = \langle +x|1\rangle = \frac{1}{\sqrt{2}} \ , \tag{26}$$

$$\langle -x|0\rangle = -\langle -x|1\rangle = \frac{1}{\sqrt{2}} \ . \tag{27}$$

The following projections can now be easily computed,

$$\hat{P}_+\hat{\rho}_0\hat{P}_+ = \frac{1}{4}|+x\rangle\langle +x| \otimes \left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right) \ , \tag{28}$$

$$\hat{P}_-\hat{\rho}_0\hat{P}_- = \frac{1}{4}|-x\rangle\langle -x| \otimes \left(|00\rangle\langle 00| - |00\rangle\langle 11| - |11\rangle\langle 00| + |11\rangle\langle 11|\right) \ , \tag{29}$$

which we can write more compactly as,

$$\hat{P}_+\hat{\rho}_0\hat{P}_+ = \frac{1}{2}|+x\rangle\langle +x| \otimes \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)\left(\frac{\langle 00| + \langle 11|}{\sqrt{2}}\right) \ , \tag{30}$$

$$\hat{P}_-\hat{\rho}_0\hat{P}_- = \frac{1}{2}|-x\rangle\langle -x| \otimes \left(\frac{|00\rangle - |11\rangle}{\sqrt{2}}\right)\left(\frac{\langle 00| - \langle 11|}{\sqrt{2}}\right) \ . \tag{31}$$

It's a good idea to express the first qubit, on which the PVM projectors have been applied, in the measurement basis; in this case the $\{|\pm x\rangle\}$ basis since the outcome of such a measurement will be in the said basis. Expressing it in the standard $\{|0\rangle, |1\rangle\}$ basis is somewhat obscuring the interpretation of the PVM. Let us also compute the traces of these projections which will serve as normalization factors and probabilities as well,

$$\text{Tr}\,(\hat{P}_+\hat{\rho}_0\hat{P}_+) = \text{Tr}\,(\hat{P}_-\hat{\rho}_0\hat{P}_-) = \frac{1}{2} \ . \tag{32}$$

This can be easily seen from Eqs. (30) and (31) since $\mathrm{Tr}\,(\hat{P}_+) = \mathrm{Tr}\,(\hat{P}_-) = 1$ and also the residual states of the second and third qubits $\left(\frac{|00\rangle \pm |11\rangle}{\sqrt{2}}\right)$ are normalized too. The measurement interpretation of the PVM is that with probability $p_+ = \mathrm{Tr}\,(\hat{P}_+\hat{\rho}_0\hat{P}_+)$, the outcome of the measurement on the first qubit is observed to be "$+x$" and the post-measurement state in this case will be,

$$\hat{\rho}_+ = \frac{\hat{P}_+\hat{\rho}_0\hat{P}_+}{\mathrm{Tr}\,(\hat{P}_+\hat{\rho}_0\hat{P}_+)} \,, \tag{33}$$

and with probability $p_- = \mathrm{Tr}\,(\hat{P}_-\hat{\rho}_0\hat{P}_-)$, the outcome observed it,

$$\hat{\rho}_- = \frac{\hat{P}_-\hat{\rho}_0\hat{P}_-}{\mathrm{Tr}\,(\hat{P}_-\hat{\rho}_0\hat{P}_-)} \,, \tag{34}$$

Okay, now we can simply state the questions asked for in the homework.

If the post-measurement state is not known to the observer, the density operator describing the GHZ will be given by,

$$\hat{\rho}_1 = p_+\hat{\rho}_+ + p_-\hat{\rho}_- \,, \tag{35}$$

$$\hat{\rho}_1 = \frac{1}{2}\,|+x\rangle\,\langle +x| \otimes \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)\left(\frac{\langle 00| + \langle 11|}{\sqrt{2}}\right) + \frac{1}{2}\,|-x\rangle\,\langle -x| \otimes \left(\frac{|00\rangle - |11\rangle}{\sqrt{2}}\right)\left(\frac{\langle 00| - \langle 11|}{\sqrt{2}}\right) . \tag{36}$$

On the other hand, for part (c.) with probability $p_+ = 1/2$ we will observe the post-measurement state to be $\hat{\rho}_+$ of Eq. (33) in which case the residual state of the second and third qubits can simply be "read-off" as,

$$|\psi_{23}^+\rangle = \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right) . \tag{37}$$

The other possibility of the measurement outcome is observing the "$-x$" outcome for the first qubit with probability $p_- = 1/2$, in which case the GHZ state is given by $\hat{\rho}_-$ of Eq. (34) and the residual state of the other two qubits will be,

$$|\psi_{23}^-\rangle = \left(\frac{|00\rangle - |11\rangle}{\sqrt{2}}\right) . \tag{38}$$

It is worth pointing out here that even though we found pure residual states for the other two qubits; in general, the residual state of sub-systems on which measurement is not performed will be mixed.

4 Thermal Spins in a B-field (5 points)

In this question, for all our matrix representations, we will exclusively work in the $\hat{\sigma}_z$ basis. The Hamiltonian of the spin 1/2 particle in the B-field is simply,

$$\hat{H} = -\frac{\gamma B}{2}\hat{\sigma}_z \equiv \begin{bmatrix} -\frac{\omega}{2} & 0 \\ 0 & \frac{\omega}{2} \end{bmatrix} \,, \tag{39}$$

where we have defined $\omega = \gamma B$ for convenience. Now since the particle is a part of a thermal ensemble at temperature $T \equiv 1/k_B\beta$ with k_B being the Boltzmann constant, its density operator is thermal,

$$\hat{\rho} = \frac{\exp(-\beta\hat{H})}{\mathrm{Tr}\,(\exp(-\beta\hat{H}))} \,. \tag{40}$$

Since we are asked to work in the $\hat{\sigma}_z$ basis in which $\hat{H}$ is diagonal, hence $\exp(-\beta\hat{H})$ will be diagonal in this basis too, given by,

$$\exp(-\beta\hat{H}) \equiv \begin{bmatrix} e^{\beta\omega/2} & 0 \\ 0 & e^{-\beta\omega/2} \end{bmatrix} , \tag{41}$$

whose trace is simply $\mathrm{Tr}\left(\exp(-\beta\hat{H})\right) = e^{\beta\omega/2} + e^{-\beta\omega/2}$ and hence,

$$\rho = \frac{1}{e^{\beta\omega/2} + e^{-\beta\omega/2}} \begin{bmatrix} e^{\beta\omega/2} & 0 \\ 0 & e^{-\beta\omega/2} \end{bmatrix} . \tag{42}$$

(b.) Now we compute the expectation values of $\hat{\vec{\sigma}}$ for the particle in this thermal ensemble. For $\hat{\sigma}_l$, the expectation value is given by,

$$\langle\hat{\sigma}_l\rangle = \mathrm{Tr}\left(\hat{\sigma}_l\hat{\rho}\right) . \tag{43}$$

It can now be seen rather straightforwardly by matrix multiplication and taking the trace that $\langle\hat{\sigma}_x\rangle = \langle\hat{\sigma}_y\rangle = 0$ and,

$$\langle\hat{\sigma}_z\rangle = \frac{e^{\beta\omega/2} - e^{-\beta\omega/2}}{e^{\beta\omega/2} + e^{-\beta\omega/2}} = \tanh\left(\frac{\beta\omega}{2}\right) \tag{44}$$

(c.) Now we can compute the average magnetization of N such particles in the thermal ensemble,

$$\vec{M} = \frac{N\gamma}{2}\left(\langle\hat{\sigma}_x\rangle\,\hat{\imath} + \langle\hat{\sigma}_y\rangle\,\hat{\jmath} + \langle\hat{\sigma}_z\rangle\,\hat{k}\right) , \tag{45}$$

where the expectations are the single particle expectations computed in part (b) above and $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ are spatial unit vectors along x, y and z directions, respectively. This gives us,

$$\vec{M} = \frac{N\gamma}{2}\tanh\left(\frac{\beta\omega}{2}\right)\hat{z} . \tag{46}$$

Now we consider the high temperature or the small β limit and Taylor expand the magnitude of $\vec{M}$ to linear order in β to get the characteristic Curie's Law $1/T$ dependence of the magnetization,

$$M = \frac{N\gamma^2 B}{4k_B}\frac{1}{T} + \mathcal{O}\left(\frac{1}{T^3}\right) . \tag{47}$$
